

# A QUIVER PRESENTATION FOR SOLOMON'S DESCENT ALGEBRA.

GÖTZ PFEIFFER

*Dedicated to the memory of Manfred Schocker (1970–2006)*

ABSTRACT. The descent algebra  $\Sigma(W)$  is a subalgebra of the group algebra  $\mathbb{Q}W$  of a finite Coxeter group  $W$ , which supports a homomorphism with nilpotent kernel and commutative image in the character ring of  $W$ . Thus  $\Sigma(W)$  is a basic algebra, and as such it has a presentation as a quiver with relations. Here we construct  $\Sigma(W)$  as a quotient of a subalgebra of the path algebra of the Hasse diagram of the Boolean lattice of all subsets of  $S$ , the set of simple reflections in  $W$ . From this construction we obtain some general information about the quiver of  $\Sigma(W)$  and an algorithm for the construction of a quiver presentation for the descent algebra  $\Sigma(W)$  of any given finite Coxeter group  $W$ .

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## 1. INTRODUCTION.

The descent algebra  $\Sigma(W)$  of a finite Coxeter group  $W$  of rank  $n$  is a remarkable  $2^n$ -dimensional subalgebra of the group algebra  $\mathbb{Q}W$ , which supports a homomorphism  $\theta$  with nilpotent kernel and commutative image in the character ring of  $W$ . Therefore,  $\Sigma(W)$  is a basic algebra, and as such it has a presentation as a quiver with relations. In this article we construct  $\Sigma(W)$  as a quotient of a subalgebra of the path algebra of the Hasse diagram of the power set of  $S$ , the set of simple reflections of  $W$ . From this construction we obtain general information about the quiver of  $\Sigma(W)$  and an algorithm, which for a given finite Coxeter group  $W$  computes a quiver presentation for the descent algebra  $\Sigma(W)$ .

Solomon [22] has defined the descent algebra  $\Sigma(W)$  in terms of distinguished coset representatives of the standard parabolic subgroups of  $W$ . Bergeron, Bergeron, Howlett and Taylor [2] have obtained a decomposition of  $\Sigma(W)$  into principal indecomposable modules. More recently, Bidigare [3] has identified  $\Sigma(W)$  with the fixed point space under the action of  $W$  on the monoid algebra of the face monoid of the hyperplane arrangement associated to the reflection representation of  $W$ . In this approach, the descent algebra  $\Sigma(W)$  is a subalgebra of a quotient of a path algebra. Brown [8] discusses this construction in the wider context of semigroups of idempotents. The face monoid algebra of the reflection arrangement of a finite Coxeter group  $W$  is called the Solomon-Tits algebra by Patras and Schocker [13]. Schocker [19] discusses the descent algebra of the symmetric group and its quiver in this context.

More results concerning the quiver of  $\Sigma(W)$  have been obtained for particular types of finite Coxeter groups, mostly for type  $A_n$ . Garsia and Reutenauer [9] have performed a very detailed analysis of the descent algebra of the symmetric group, and described its quiver in terms of restricted partition refinement. Atkinson [1] has determined the Loewy length of the descent algebra of the symmetric group. Bonnafé and Pfeiffer [6] have determined the Loewy length of  $\Sigma(W)$  for the other types of irreducible finite Coxeter groups with the exception of type  $D_n$  for  $n$  odd. An argument which settles this case has been put forward by Saliola [16], based on his investigation of Bidigare's geometric setting [17].

In this article, we present an alternative approach to  $\Sigma(W)$ . We construct a quiver with relations for  $\Sigma(W)$  in three steps. The point of departure is the path algebra  $A$  of the Hasse diagram of the power set  $\mathcal{P}(S)$  of a finite set  $S$ , partially ordered by reverse inclusion. Then we use a partial action of  $W$  on  $\mathcal{P}(S)$  to exhibit a subalgebra of  $A$ , and a quiver presentation for it. Finally, a quotient of this subalgebra, formed with the help of a difference operator on  $A$ , is shown to be isomorphic to  $\Sigma(W)$ .

This article is organized as follows. In Section 2 we recall the definition of the descent algebra in terms of the distinguished coset representatives of parabolic subgroups of  $W$  and some combinatorial properties of these transversals. Section 3 introduces quivers and their path algebras, and shows how monoid actions, in particular of a free monoid, produce examples of quivers. In Section 4, the conjugation action of  $W$  on its parabolic subgroups is described as an action of a free monoid on the standard parabolic subgroups of  $W$ . In Section 5 we obtain the Hasse diagram of the power set of  $S$  from the take-away action of the free monoid  $S^*$ . The paths in this particular quiver are called alleys and they form a basis of a path algebra  $A$ . Prefixes and suffixes of paths define in a natural way two rooted forests on the set of all alleys. In Section 6, we apply the conjugation action from Section 4 to the alleys of Section 5. An orbit of alleys is called a street and the streets (identified with the sums of their elements) form a basis of a subalgebra  $\Xi$  of  $A$ . Prefixes and suffixes of alleys induce two rooted forests on the set of all streets, which in particular decompose  $\Xi$  into projective indecomposable modules. We furthermore conjecture that  $\Xi$  is a path algebra. In Section 7, a difference operator  $\Delta$  on  $A$  is used to map  $\Xi$  surjectively onto the grade 0 component  $A_0$  of  $A$ . In Section 8, we use the difference operator to define a matrix representation of  $\Xi$  on  $A_0$ . In Section 9, we prove in Theorem 9.3 a key result about right multiplication in  $\Sigma(W)$ . We then identify  $A_0$  with the descent algebra  $\Sigma(W)$ , and show as our Main Theorem 9.5 that with this identification  $\Delta$  becomes an anti-homomorphism from  $\Xi$  onto  $\Sigma(W)$ . In Section 10, we derive some properties of the quiver of  $\Sigma(W)$  from this construction. Finally, in Section 11, we present an algorithm which computes, for a given finite Coxeter group  $W$ , a quiver presentation for  $\Sigma(W)$ . For each of the series  $A_n$ ,  $B_n$ , and  $D_n$  of irreducible finite Coxeter groups, we state some general properties of the quiver of  $\Sigma(W)$  and give one example of a quiver presentation.

Throughout, we use the symmetric group on 4 points for the purpose of illustration. The constructions, however, work for all types of finite Coxeter groups. Concrete

results for particular types will be the subject of subsequent articles. Computer implementations of data structures corresponding to some of the combinatorial and algebraic objects introduced here have helped us to produce the examples and figures, and to verify conjectured theorems in many cases. They are available in the form of the GAP [20] package **ZigZag** [14], which is based on the **CHEVIE** [10] package for finite Coxeter groups and Iwahori–Hecke algebras.

## 2. DESCENTS AND PARABOLIC TRANSVERSALS.

In this section, notation and some basic concepts are introduced, mostly following Geck and Pfeiffer [11]. Let  $W$  be a finite Coxeter group, generated by a set  $S$  of simple reflections. Let  $\ell: W \rightarrow \mathbb{N}_0$  be the usual length function on  $W$ . The (left) *descent set* of an element  $w \in W$  is the set

$$(2.1) \quad \mathcal{D}(w) = \{s \in S : \ell(sw) < \ell(w)\}.$$

For each subset  $J \subseteq S$ , the subgroup  $W_J = \langle J \rangle$  is called a (standard) *parabolic subgroup* of  $W$ , and the set

$$(2.2) \quad X_J = \{w \in W : \mathcal{D}(w) \cap J = \emptyset\}$$

is a transversal of the right cosets  $W_J w$  of the parabolic subgroup  $W_J$  in  $W$ , consisting of the elements of minimal length in each coset. For a fixed subset  $J \subseteq S$ , each element  $w \in W$  can be written as a product  $w = u \cdot x$  for unique elements  $u \in W_J$  and  $x \in X_J$ . A product  $w_1 w_2 \cdots w_k$  of elements  $w_1, w_2, \dots, w_k \in W$  is called *reduced* if

$$(2.3) \quad \ell(w_1 w_2 \cdots w_k) = \ell(w_1) + \ell(w_2) + \cdots + \ell(w_k).$$

If this is the case, we sometimes write a product like  $w_1 w_2 w_3$  as  $w_1 \cdot w_2 \cdot w_3$  in order to emphasize the fact that the product is reduced. For example, the product  $u \cdot x$  of an element  $u \in W_J$  and a coset representative  $x \in X_J$  is reduced. The *longest element* of the parabolic subgroup  $W_J$  is denoted by  $w_J$ , the longest element of  $W$  also by  $w_0$ . The elements  $w_J$  are involutions. The quotient  $w_J^{-1} w_0 = w_J w_0$  is the unique longest element of the transversal  $X_J$ .

The *descent algebra* of  $W$  is defined as the subspace  $\Sigma(W)$  of the group algebra  $\mathbb{Q}W$  spanned by the sums

$$(2.4) \quad x_J = \sum_{x \in X_J^{-1}} x \in \mathbb{Q}W$$

over the sets  $X_J^{-1} = \{x^{-1} : x \in X_J\}$  of *left* coset representatives of  $W_J$  in  $W$ , for  $J \subseteq S$ . By Solomon's Theorem [22], this subspace is in fact a subalgebra of  $\mathbb{Q}W$  with structure constants as in equation (2.8) below. For  $J, K \subseteq S$ , one further defines

$$(2.5) \quad X_{JK} = X_J \cap X_K^{-1} \quad \text{and} \quad X_J^K = X_J \cap W_K.$$

The set  $X_{JK}$  is a transversal of the double cosets of  $W_J$  and  $W_K$  in  $W$ . A parabolic transversal  $X_J$  can in many ways be described in terms of other transversals, or as a set of prefixes. Here, an element  $u \in W$  is called a *prefix* of  $w \in W$  if  $l(w) = l(u) + l(u^{-1}w)$ . In that case we write  $u \leq w$ . The partial order defined in this way on  $W$  is sometimes called the *weak Bruhat order* on  $W$ .

**2.1. Proposition.** *Let  $J, K \subseteq S$ . Then*

- (i)  $X_J = X_J^K \cdot X_K$  if  $J \subseteq K$ ;
- (ii)  $X_J = dX_K$  if  $d \in X_J$  and  $K$  are such that  $J^d = K$ ;
- (iii)  $X_J = \bigsqcup_{d \in X_{JK}} d \cdot X_{J^d \cap K}^K$ ;
- (iv)  $X_J = \{w \in W : w \leq w_J w_0\}$ . Thus,  $w \in X_J$  whenever  $w \leq x$  for some  $x \in X_J$ .

*Proof.* [11, (2.1.5), (2.1.8), (2.1.9), and (2.2.1)]. □

For subsets  $J, K, L \subseteq S$ , we furthermore define

$$(2.6) \quad X_{JKL} = \{x \in X_{JK} : J^x \cap K = L\}.$$

The cardinalities

$$(2.7) \quad a_{JKL} = |X_{JKL}|$$

are the structure constants of the descent algebra: according to Solomon [22], for all  $J, K \subseteq S$ ,

$$(2.8) \quad x_J x_K = \sum_{d \in X_{JK}} x_{J^d \cap K} = \sum_{L \subseteq S} a_{JKL} x_L.$$

Denote by  $\theta$  the linear map from  $\Sigma(W)$  into the character ring of  $W$  (over  $\mathbb{Q}$ ) which assigns, for  $J \subseteq S$ , to  $x_J$  the permutation character

$$(2.9) \quad \theta(x_J) = 1_{W_J}^W,$$

of the action of  $W$  on the cosets of the parabolic subgroup  $W_J$  in  $W$ . Then, according to Solomon [22],  $\theta$  is a homomorphism of algebras with commutative image and nilpotent kernel. It follows that the descent algebra  $\Sigma(W)$  is a basic algebra, and as such it has a presentation as a quiver with relations.

### 3. QUIVERS, PATH ALGEBRAS AND MONOID ACTIONS.

A *quiver* is a directed multigraph  $Q = (V, E)$  consisting of a vertex set  $V$  and an edge set  $E$ , together with two maps  $\iota, \tau: E \rightarrow V$ , assigning to each edge  $e \in E$  a *source* (or *initial vertex*)  $\iota(e) \in V$  and a *target* (or *terminal vertex*)  $\tau(e) \in V$ . A *path of length*  $\ell(\mathbf{a}) = l$  in  $Q$  is a pair

$$(3.1) \quad \mathbf{a} = (v; e_1, e_2, \dots, e_l)$$

consisting of a source  $v \in V$  and a sequence of  $l$  edges  $e_1, e_2, \dots, e_l \in E$  such that  $\iota(e_1) = v$  and  $\iota(e_i) = \tau(e_{i-1})$  for  $i = 2, \dots, l$ .

Let  $\mathcal{A}$  be the set of all paths in  $Q$  and let

$$(3.2) \quad \mathcal{A}_l = \{\mathbf{a} \in \mathcal{A} : \ell(\mathbf{a}) = l\}$$

be the set of paths of length  $l$ . We denote by  $\emptyset$  the unique sequence of length 0 and identify a vertex  $v \in V$  with the path  $(v; \emptyset) \in \mathcal{A}_0$ . We also identify an edge  $e \in E$  with the path  $(\iota(e); e) \in \mathcal{A}_1$ . The following properties of path concatenation are obvious.

**3.1. Proposition.** *The set  $\mathcal{A} = \bigsqcup_{l \geq 0} \mathcal{A}_l$  together with the partial multiplication defined as*

$$(v; e_1, \dots, e_l) \circ (v'; e'_1, \dots, e'_{l'}) = (v; e_1, \dots, e_l, e'_1, \dots, e'_{l'}),$$

*provided that  $\tau(e_l) = v'$ , is (the set of morphisms of) a category with object set  $\mathcal{A}_0$ . The category  $\mathcal{A}$  is the free category generated by the quiver  $Q$ . Every path  $\mathbf{a} \in \mathcal{A}$  of length  $\ell(\mathbf{a}) > 0$  is a unique product of elements in the set  $\mathcal{A}_1$ .*

The category  $\mathcal{A}$  of all paths in  $Q$  can be used as formal basis of a vector space. For  $l \geq 0$ , let  $\mathbf{A}_l = \mathbb{Q}[\mathcal{A}_l]$ , the  $\mathbb{Q}$ -vector space with basis  $\mathcal{A}_l$ . The *path algebra*  $\mathbf{A}$  of the quiver  $Q$  is defined as

$$(3.3) \quad \mathbf{A} = \mathbb{Q}[\mathcal{A}] = \bigoplus_{l \geq 0} \mathbf{A}_l,$$

where  $\mathbf{a} \circ \mathbf{a}' = 0$  if the product  $\mathbf{a} \circ \mathbf{a}'$  is not defined in  $\mathcal{A}$ , and otherwise multiplication is extended by linearity from  $\mathcal{A}$ . The path algebra  $\mathbf{A}$  is a graded algebra, since we have  $A_l \circ A_k \subseteq A_{l+k}$ , for all  $l, k \geq 0$ .

**3.2. Monoid Actions.** A good source for examples of quivers and categories are monoid actions. Suppose  $\mathbf{M}$  is a monoid acting on a set  $X$  via  $(x, m) \mapsto x.m$ , then the set  $X \times \mathbf{M}$  together with the partial multiplication

$$(3.4) \quad (x, m) \circ (x', m') = (x, mm'),$$

whenever  $x, x' \in X$  and  $m, m' \in \mathbf{M}$  are such that  $x.m = m'$ , is a category with object set  $X$ .

If  $\mathbf{M}$  is generated by a set  $S \subseteq \mathbf{M}$ , then the *action graph* defined as the directed multigraph with vertex set  $X$  and edge set  $X \times S$  is a quiver with  $\iota(x, s) = x$  and  $\tau(x, s) = x.s$  for all  $(x, s) \in X \times S$ . If  $\mathbf{M}$  is the free monoid  $S^*$ , then the category  $X \times \mathbf{M}$  is the free category generated by the action graph  $(X, X \times S)$ .

In the following two sections, we consider two different, but related, examples of actions of the free monoid  $S^*$  on the power set  $\mathcal{P}(S)$  of a finite set  $S$ . In section 6, we apply one action to the path algebra arising from the other.

#### 4. SHAPES.

Let  $W$  be the finite Coxeter group from Section 2 and let  $S \subseteq W$  be its set of simple reflections. Here we regard  $W$  as a quotient of the free monoid  $S^*$  consisting of all words over the alphabet  $S$ . The empty word will be denoted by  $\emptyset$ ; the identity element of  $W$  by  $\text{id}$ . Beyond that, we make no notational effort to distinguish a word in  $S^*$  from a product of simple reflections in  $W$ .

The conjugation action of  $W$  on itself induces a conjugation action of  $W$  on its subsets which partitions the power set  $\mathcal{P}(S) \subseteq \mathcal{P}(W)$  into classes of  $W$ -conjugate subsets. We write

$$(4.1) \quad A \sim B$$

if  $A, B \subseteq W$  are such that  $B = A^w$  for some  $w \in W$  and call the class

$$(4.2) \quad [J] = \{K \subseteq S : K \sim J\}$$

of a subset  $J \subseteq S$  the *shape* of  $J$  in  $W$ . Moreover, we denote by

$$(4.3) \quad \Lambda = \{[J] : J \subseteq S\}$$

the set of all shapes of  $W$ . The shapes of  $W$  correspond to the conjugacy classes of parabolic subgroups of  $W$ , since, for  $J, K \subseteq S$ , the parabolic subgroup  $W_K$  is a conjugate of  $W_J$  if and only if  $K$  is a conjugate of  $J$  [11, (2.1.13)].

**4.1. Example.** Suppose  $W$  is the symmetric group  $\text{Sym}(n+1)$  of degree  $n+1$ . Every parabolic subgroup of  $W$  is a direct product of symmetric groups, whose degrees form a partition of  $n+1$ , if fixed points are counted as factors of degree 1. Two parabolic subgroups are conjugate in  $W$  if and only if the corresponding partitions are the same. In this way the shapes of a Coxeter group of type  $A_n$  correspond to the partitions of  $n+1$ .

In a similar way, the shapes of a Coxeter group of type  $B_n$  correspond to the partitions of all  $m \in \{0, \dots, n\}$  [11, (2.3.10)]. The shapes of a Coxeter group of type  $D_n$  correspond to the partitions of all  $m \in \{0, \dots, n-2\}$ , and the partitions of  $n$ , with two copies of each even partition of  $n$  [11, (2.3.13)].

In order to decide whether a parabolic subgroup  $W_K$  of  $W$  is a conjugate  $W_J^x$  of a parabolic subgroup  $W_J$ , it clearly suffices to consider elements  $x \in X_J$  which conjugate  $J \subseteq S$  to a subset of  $S$ . For  $J \subseteq S$  we denote

$$(4.4) \quad X_J^\sharp = \{x \in X_J : J^x \subseteq S\} = \bigsqcup_{K \sim J} X_{JKK}.$$

Certainly  $\text{id} \in X_J^\sharp$  for all  $J \subseteq S$ . And it is easy to see that  $X_L^\sharp \subseteq X_J^\sharp$  for all  $J \subseteq L \subseteq S$ . In general, the elements of  $X_J^\sharp$  can be described as reduced products of certain longest coset representatives. Given  $J \subseteq S$  and  $s \in S$ , we let  $L = J \cup \{s\}$  and denote

$$(4.5) \quad \omega(J, s) = w_J w_L \in X_J^\sharp,$$

the longest coset representative of  $W_J$  in  $W_L$ . Clearly, if  $s \in J$  then  $\omega(J, s) = \text{id}$ . Note that

$$(4.6) \quad \omega(J, s)^{-1} = \omega(J^{w_L}, s^{w_L}).$$

**4.2. Lemma.** *Let  $J \subseteq S$  and let  $x \in X_J^\sharp$ .*

- (i)  $x^{-1} \in X_{J^x}^\sharp$ .



- (ii) If  $J$  is a maximal subset of  $S$  then  $x = \text{id}$  or  $x = w_J w_0$ .
- (iii) If  $s \in \mathcal{D}(x)$  then  $\omega(J, s)$  is a prefix of  $x$ . Moreover,  $\omega(J, s)^{-1}x \in X_K^\sharp$ , where  $K = J^{\omega(J, s)}$ .
- (iv) There exist elements  $s_1, \dots, s_r \in S$  such that

$$x = \omega(J_1, s_1) \cdot \omega(J_2, s_2) \cdots \omega(J_r, s_r)$$

is a reduced product, where  $J_1 = J$  and  $J_{k+1} = J_k^{\omega(J_k, s_k)}$  for  $1 \leq k < r$ .

*Proof.* (i) If  $J^x = K \subseteq S$  then clearly  $K^{x^{-1}} = J \subseteq S$ . It remains to show that  $x^{-1} \in X_K = \{w \in W : K \cap \mathcal{D}(w) = \emptyset\}$ . But if there exists an element  $s \in K \cap \mathcal{D}(x^{-1})$  then  $s^{x^{-1}} = xsx^{-1} \in J \cap \mathcal{D}(x)$ , contradicting  $x \in X_J$ .

(ii) [11, (2.3.2)].

(iii) Let  $L = J \cup \{s\}$ . Using Proposition 2.1(i), we can write  $x = x_1 \cdot x_2$  for (unique) elements  $x_1 \in X_J^L$  and  $x_2 \in X_L$ . Clearly  $x_1 \in X_J^\sharp$  and  $x_1 \neq \text{id}$  since  $s \in \mathcal{D}(x_1)$ . With (ii), this forces  $x_1 = w_J w_L$ , since  $J$  is a maximal subset of  $L$ .

(iv) follows by induction on  $\ell(x)$  from (iii). □

The preceding lemma motivates the following definition of an action of the free monoid  $S^*$  on the power set  $\mathcal{P}(S)$ . For  $J \subseteq S$  and  $s \in S$  we set

$$(4.7) \quad J.s = J^{\omega(J, s)}.$$

Most of the following properties are obvious.

**4.3. Lemma.** *Let  $J \subseteq S$ ,  $s \in S$  and  $L = J \cup \{s\}$ . Then*

- (i)  $J.s = J$  if  $s \in J$ ;
- (ii)  $J.ss = J.s$ ;
- (iii)  $s^{w_L} \in S$  and  $J.ss^{w_L} = J$ .

*Proof.* (ii) If  $J.s \neq J$  then  $s \in J.s$ . (iii) See equation (4.6). □

Lemma 4.3(iii) shows that all the effects of  $S^*$  on  $\mathcal{P}(S)$  can be undone. Therefore, the  $S^*$ -orbits on  $\mathcal{P}(S)$  form a partition of  $\mathcal{P}(S)$ . In fact, as a consequence of

Lemma 4.2(iv), these orbits coincide with the shapes of  $W$ . The following theorem states some known properties of normalizers of parabolic subgroups in terms of the conjugation action of  $S^*$  on the power set  $\mathcal{P}(S)$ .

**4.4. Theorem** (Howlett). *Suppose  $S^*$  acts on  $\mathcal{P}(S)$  as defined above.*

(i) *The orbits of  $S^*$  on  $\mathcal{P}(S)$  form a partition of  $\mathcal{P}(S)$  and, for all  $J \subseteq S$ , the  $S^*$ -orbit of  $J$  coincides with the shape  $[J]$  in  $W$ .*

(ii) *The stabilizer*

$$N_J = \{x \in X_J : J^x = J\} = X_{JJ}$$

*of  $J$  is a complement of  $W_J$  in its normalizer  $N_W(W_J) = W_J \rtimes N_J$ .*

*Proof.* [12]; see also [11, Theorem 2.3.3 and Proposition 2.1.15].  $\square$

According to Section 3.2, this action gives rise to a category  $\mathcal{P}(S) \times S^*$ . We will ignore the obviously trivial elements  $(J, s)$  where  $s \in J$  (see Lemma 4.3(i)) and define the *action graph* of  $S^*$  on  $\mathcal{P}(S)$  as the quiver with edge set

$$(4.8) \quad \{(J, s) : J \subseteq S, s \in S - J\},$$

where  $\iota(J, s) = J$  and  $\tau(J, s) = J.s$ . We then define a category  $\mathcal{C}$  as the subcategory of  $\mathcal{P}(S) \times S^*$  which is generated as a free category by this action graph.

**4.5. Example.** Figure 1 illustrates the action graph and the category  $\mathcal{C}$  for the type  $A_3$  generated by a set  $S = \{1, 2, 3\}$  of simple reflections, where 1 commutes with 3. Here the vertices are the subsets of  $S$ , written with braces and commas omitted. Multiple labels (like 1, 2, 3) on arrows indicate multiple arrows, one for each label.

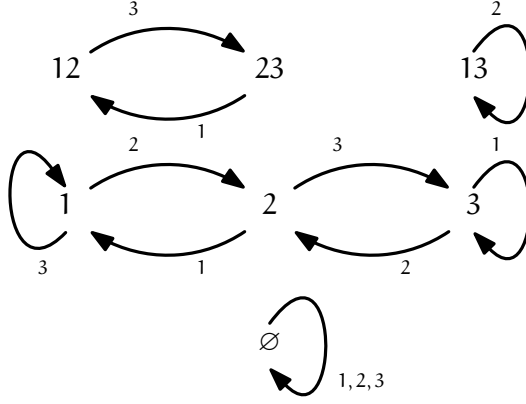
The fact that the effects of the monoid  $S^*$  on any subset  $J \subseteq S$  can be undone manifests itself further in the form of a closely related groupoid. Note that if  $J \sim K$  then  $X_{JKK} = \{x \in X_J : J^x = K\}$ . From Proposition 2.1(ii) it follows that, if  $J \sim K \sim L$  then  $dX_{KLL} = X_{JLL}$  for all  $d \in X_{JKK}$ , and hence that  $X_{JKK}X_{KLL} = X_{JLL}$ . Recall that  $X_J^\# = \bigsqcup_{K \sim J} X_{JKK}$ . Then the set of pairs

$$(4.9) \quad \mathcal{G} = \{(J, x) : J \subseteq S, x \in X_J^\#\}$$

forms a category with respect to the partial multiplication

$$(4.10) \quad (J, x)(K, y) = (J, xy)$$

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FIGURE 1.  $S^*$  acting on  $\mathcal{P}(S)$  of type  $A_3$ .

if  $J, K \subseteq S$ ,  $x \in X_J^\#$  and  $y \in X_K^\#$  are such that  $J^x = K$ . Each pair  $(J, x) \in \mathcal{G}$  has an inverse  $(J^x, x^{-1}) \in \mathcal{G}$ , since by Lemma 4.2(i)  $x \in X_J^\#$  implies  $x^{-1} \in X_{J^x}^\#$ . Therefore, the category  $\mathcal{G}$  is in fact a groupoid.

There exists a unique functor  $\omega$  from the category  $\mathcal{P}(S) \times S^*$  to the group  $W$ , regarded as a one-object category, extending the map  $\omega: \mathcal{P}(S) \times S \rightarrow W$  as defined in equation (4.5). Then,

$$(4.11) \quad J \cdot m = J^{\omega(J, m)}$$

for all  $J \in \mathcal{P}(S)$  and all  $m \in S^*$ , and conjugation by  $\omega(J, m)$  induces a bijection

$$(4.12) \quad s \mapsto s^{\omega(J, m)}$$

from  $J$  to  $J \cdot m$ .

**4.6. Proposition.** *The map  $(J, m) \mapsto (J, \omega(J, m))$  together with the identity map on the object set  $\mathcal{P}(S)$  is a functor from the category  $\mathcal{P}(S) \times S^*$  onto the groupoid  $\mathcal{G}$ .*

*Proof.* Let  $J \subseteq S$ . We only need to show that  $\omega(J, m) \in X_J$  for all  $m \in S^*$ . If  $m = \emptyset$  then  $\omega(J, m) = \text{id}$  and there is nothing to show. Otherwise  $m = sm'$  for some  $s \in S$  and  $m' \in S^*$ . Let  $d = \omega(J, s)$ . Then  $\omega(J, m) = d\omega(J^d, m')$ , where  $\omega(J^d, m') \in X_{J^d}$ , by induction on the length of  $m'$ . By Proposition 2.1,  $dX_{J^d} = X_J$ , whence  $\omega(J, m) = d\omega(J^d, m') \in X_J$ .  $\square$

Brink and Howlett [7] have given a presentation of  $\mathcal{G}$  as a quotient of the free category  $\mathcal{C}$  in terms of the generating set

$$(4.13) \quad \{(J, \omega(J, s)) : J \subseteq S, s \in S - J\}$$

corresponding to the edges of the action graph. Before we formulate the relations, we introduce the notion of a reduced expression for elements of  $\mathcal{G}$ .

We call  $(J, m) \in \mathcal{C}$  with  $m = s_1 s_2 \cdots s_k \in S^*$  a *reduced expression* for  $(K, x) \in \mathcal{G}$  if  $(J, \omega(J, m)) = (K, x)$  and

$$(4.14) \quad x = \omega(J_1, s_1) \cdot \omega(J_2, s_2) \cdots \omega(J_k, s_k)$$

is a reduced product in  $W$  in the sense of equation (2.3), where  $J_i = J.s_1 s_2 \cdots s_{i-1}$ , for  $i = 1, \dots, k$ . In particular, for  $J \subseteq S$ , the pair  $(J, \emptyset) \in \mathcal{C}$  is the unique reduced expression for  $(J, \text{id}) \in \mathcal{G}$ .

By Lemma 4.2(iv), each element  $(J, x) \in \mathcal{G}$  has a reduced expression. Suppose  $J \subseteq L \subseteq M \subseteq S$ . Then clearly  $w_L w_M \in X_J^\sharp$  and  $(J, w_L w_M) \in \mathcal{G}$  has a reduced expression. The following result by Brink and Howlett [7] is concerned with a situation where this reduced expression is unique.

**4.7. Proposition.** *Let  $J \subseteq S$  and  $s, t \in S - J$  be such that  $s \neq t$ . Let  $L = J \cup \{s\}$  and let  $w \in X_J^\sharp$  be a prefix of  $\omega(L, t)$ . Then the element  $(J, w) \in \mathcal{G}$  has a unique reduced expression. In particular, the reduced expression for  $(J, \omega(L, t)) \in \mathcal{G}$  is unique.*

*Proof.* Without loss of generality we may assume that  $L \cup \{t\} = S$ . Then, by Proposition 2.1(iv),  $w$  is a prefix of  $\omega(L, t)$  if and only if  $w \in X_L$ .

The claim is certainly true for  $w = \text{id}$ . Otherwise,  $w \in X_J^\sharp \cap X_L$  is such that  $l(w) > 0$ . It follows that  $\mathcal{D}(w) = \{t\}$  and hence, using Lemma 4.2(iii), that each reduced expression for  $(J, w) \in \mathcal{G}$  begins with  $(J, t) \in \mathcal{C}$ .

Let  $K = J \cup \{t\}$ . A straightforward comparison of lengths shows that the product

$$(4.15) \quad (\omega(J, t)^{-1} \omega(L, t)) \cdot \omega(J, s)^{\omega(L, t)} = ((w_J w_K)^{-1} w_L w_0) \cdot (w_J w_L)^{w_L w_0} \\ = w_K w_0 = \omega(K, s)$$

is reduced in  $W$ . Therefore the element

$$(4.16) \quad w' = \omega(J, t)^{-1} w \in X_{J, t}^\sharp$$

is a prefix of  $\omega(K, s)$ . By induction on the length  $\ell(w)$ , the pair  $(J.t, w') \in \mathcal{G}$  has a unique reduced expression. This shows that

$$(4.17) \quad (J, w) = (J, \omega(J, t)) \cdot (J.t, w') \in \mathcal{G}$$

has a unique reduced expression.  $\square$

The last result implies that  $(J, w_J w_M)$ , where  $M = J \cup \{s, t\}$ , has exactly two reduced expressions,  $(J, sm_L)$  and  $(J, tm_K)$ , where  $m_L, m_K \in S^*$  are such that  $(J.s, m_L) \in \mathcal{C}$  is the unique reduced expression for  $(J.s, \omega(L, t)) \in \mathcal{G}$  and  $(J.t, m_K) \in \mathcal{C}$  is the unique reduced expression for  $(J.t, \omega(K, s)) \in \mathcal{G}$ . These two reduced expressions for the same element give rise to a relation

$$(4.18) \quad (J, sm_L) \equiv (J, tm_K).$$

It is a remarkable fact, that these relations together with relations of the form

$$(4.19) \quad (J, ss^{w_L}) \equiv (J, \emptyset),$$

which are consequences of (4.6), are sufficient for a presentation of the groupoid  $\mathcal{G}$  as a quotient of the free category  $\mathcal{C}$ .

**4.8. Theorem** (Brink–Howlett). *Let  $\equiv$  be the congruence generated by all the relations of the forms (4.19) and (4.18) in  $\mathcal{C}$ . Then  $\mathcal{C}/\equiv$  is isomorphic to  $\mathcal{G}$ .*

## 5. ALLEYS.

In this section we introduce a particular quiver  $Q$  on the vertex set  $\mathcal{P}(S)$ , whose paths will be called alleys. The quiver  $Q$  arises from the take-away action of the free monoid  $S^*$  on  $\mathcal{P}(S)$  defined by  $(L, s) \mapsto L - \{s\}$ . We exhibit various structural properties of the category  $\mathcal{A}$  of all alleys and of the corresponding path algebra  $A$  of the quiver  $Q$ . There are two natural partial orders on  $\mathcal{A}$ , which turn the set  $\mathcal{A}$  into a rooted twofold forest. In subsequent sections, the  $S^*$ -action on  $\mathcal{P}(S)$  from Section 4 will be extended to an  $S^*$ -action on  $\mathcal{A}$  and a difference operator  $\delta$  will be defined on the graded algebra  $A$ .

Let  $S$  be a finite set. For  $L \subseteq S$  and  $s \in S$  denote

$$(5.1) \quad L_s = L - \{s\}.$$

The map  $\mathcal{P}(S) \times S \rightarrow \mathcal{P}(S)$ ,  $(L, s) \mapsto L_s$ , defines an action of the free monoid  $S^*$  on the power set  $\mathcal{P}(S)$ . Ignoring the obviously trivial edges  $(L, s)$  where  $s \in S - L$ , the action graph of this action is the Hasse diagram  $Q$  of the power set  $\mathcal{P}(S)$ , partially ordered by reverse inclusion. For a subset  $L \subseteq S$  and pairwise different elements  $s, t, \dots \in L$  denote by  $(L; s, t, \dots)$  the unique path with vertices

$$(5.2) \quad L, L - \{s\}, L - \{s, t\}, \dots, L - \{s, t, \dots\}$$

in  $Q$ , deviating slightly from the notation for quivers in equation (3.1). As before in Section 3, we denote by  $\mathcal{A}$  the category of all paths in  $Q$ , and by  $A = \mathbb{Q}[\mathcal{A}]$  the path algebra of the quiver  $Q$ . We call the path  $\mathbf{a} = (L; s, t, \dots) \in \mathcal{A}$  an *alley* from its *source*  $\iota(\mathbf{a}) = L$  to its *target*  $\tau(\mathbf{a}) = L - \{s, t, \dots\}$ . The elements of the sequence  $(s, t, \dots)$  are the *segments* of  $\mathbf{a}$ , and the *length*  $\ell(\mathbf{a})$  of  $\mathbf{a}$  is the number of segments, i.e.,  $\ell(\mathbf{a}) = \#\{s, t, \dots\}$ .

5.1. **Example.** Figure 2 shows the quiver  $Q$  for  $S = \{1, 2, 3\}$ . As in Figure 1, the vertices of  $Q$  are the subsets of  $S$ , written without punctuation.

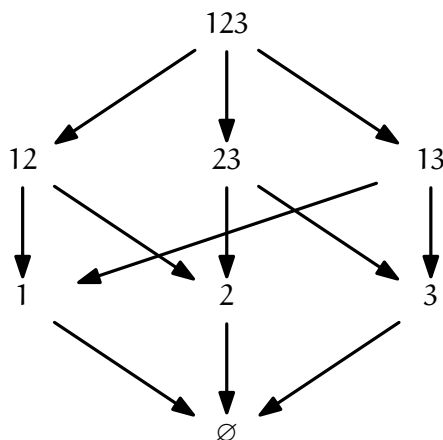


FIGURE 2. The Alley Quiver  $Q$  for  $S = \{1, 2, 3\}$ .

5.2. **Counting Alleys.** Let  $n = |S|$ . For  $0 \leq l \leq n$  there are  $\frac{2^l}{l!} n!$  alleys of length  $n - l$ , since there are  $\frac{n!}{l!}$  ways to choose  $n - l$  segments and  $2^l$  ways to choose a target from the remaining  $l$  elements of  $S$ . Hence there are

$$(5.3) \quad |\mathcal{A}| = n! \sum_{l=0}^n \frac{2^l}{l!}$$

alleys in total. The values of  $|\mathcal{A}|$  for  $n \leq 9$  are as follows.

$$(5.4) \quad \begin{array}{c|cccccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline |\mathcal{A}| & 1 & 3 & 10 & 38 & 168 & 872 & 5296 & 37200 & 297856 & 2681216 \end{array}$$

This is sequence number [A010842](#) in Sloane's online encyclopedia of integer sequences [21], which has exponential generating function  $e^{2x}/(1-x)$ .

**5.3. Partial Order.** The set  $\mathcal{A}$  of all alleys is in two ways partially ordered as follows. Let  $\mathbf{a}, \mathbf{a}' \in \mathcal{A}$ . We say that  $\mathbf{a}'$  is a *prefix* of  $\mathbf{a}$ , and write

$$(5.5) \quad \mathbf{a}' \preceq_{\pi} \mathbf{a},$$

if  $\mathbf{a} = \mathbf{a}' \circ \mathbf{a}''$  for some  $\mathbf{a}'' \in \mathcal{A}$ . By the unique factorization property of Proposition 3.1, each  $\mathbf{a} = (L; s_1, \dots, s_l) \in \mathcal{A}$  of length  $\ell(\mathbf{a}) = l > 0$  has a unique longest nontrivial prefix

$$(5.6) \quad \pi(\mathbf{a}) = (L; s_1, \dots, s_{l-1}).$$

Thus  $\mathcal{A}$  is a forest of rooted trees with roots  $L \subseteq S$ , where  $\pi(\mathbf{a})$  is the parent of  $\mathbf{a}$  if  $\ell(\mathbf{a}) > 0$ . The alley  $\mathbf{a}$  lies in the  $\pi$ -tree with root  $L$  if and only if  $\iota(\mathbf{a}) = L$ . The  $\pi$ -children of an alley  $\mathbf{a} = (L; s_1, s_2, \dots, s_l)$  are the alleys  $(L; s_1, s_2, \dots, s_{l+1})$  with  $s_{l+1} \in L - \{s_1, \dots, s_l\}$ . The alley  $\mathbf{a} = (L; s_1, s_2, \dots, s_l)$  is a  $\pi$ -leaf if  $L = \{s_1, \dots, s_l\}$ . Note that

$$(5.7) \quad \iota(\mathbf{a}) = \pi^{\ell(\mathbf{a})}(\mathbf{a})$$

for all  $\mathbf{a} \in \mathcal{A}$ .

We furthermore say that  $\mathbf{a}'$  is a *suffix* of  $\mathbf{a}$ , or that  $\mathbf{a}$  *ends in*  $\mathbf{a}'$ , and write

$$(5.8) \quad \mathbf{a} \succeq_{\sigma} \mathbf{a}'$$

if  $\mathbf{a} = \mathbf{a}'' \circ \mathbf{a}'$  for some  $\mathbf{a}'' \in \mathcal{A}$ . By the unique factorization property of Proposition 3.1, each  $\mathbf{a} = (L; s, t, \dots) \in \mathcal{A}$  of length  $\ell(\mathbf{a}) > 0$  has a unique longest nontrivial suffix

$$(5.9) \quad \sigma(\mathbf{a}) = (L_s; t, \dots).$$

Thus  $\mathcal{A}$  also is a forest of rooted trees with roots  $L \subseteq S$ , where  $\sigma(\mathbf{a})$  is the parent of  $\mathbf{a}$  if  $\ell(\mathbf{a}) > 0$ . Here the alley  $\mathbf{a}$  lies in the  $\sigma$ -tree with root  $L$  if and only if  $\tau(\mathbf{a}) = L$ .

The  $\sigma$ -children of an alley  $\mathbf{a} = (L; s, t, \dots)$  are the alleys  $(L \cup \{r\}; r, s, t, \dots)$  with  $r \in S - L$ . The alley  $\mathbf{a} = (L; s, t, \dots)$  is a  $\sigma$ -leaf if  $L = S$ . Note that

$$(5.10) \quad \tau(\mathbf{a}) = \sigma^{\ell(\mathbf{a})}(\mathbf{a})$$

for all  $\mathbf{a} \in \mathcal{A}$ .

Hence  $\mathcal{A}$  together with the two partial orders  $\preceq_\pi$  and  $\succeq_\sigma$  forms a rooted twofold forest where the subsets  $L \subseteq S$  serve as roots for both forests. In terms of these forests, the product of two alleys  $\mathbf{a}, \mathbf{a}' \in \mathcal{A}$  can be described as the intersection of the  $\pi$ -subtree with root  $\mathbf{a}$  and the  $\sigma$ -subtree with root  $\mathbf{a}'$ , provided that  $\tau(\mathbf{a}) = \iota(\mathbf{a}')$ ; in that case  $\tau(\mathbf{a}) = \iota(\mathbf{a}')$  is the only subset  $L \subseteq S$  with  $\mathbf{a} \succeq_\sigma L \preceq_\pi \mathbf{a}'$ , otherwise there is no such subset at all. In any case, this argument proves the following result.

**5.4. Proposition.** *Let  $\mathbf{a}, \mathbf{a}' \in \mathcal{A}$ . Then*

$$\mathbf{a} \circ \mathbf{a}' = \sum_{\substack{L \subseteq S, \mathbf{a}'' \in \mathcal{A} \\ \mathbf{a} \succeq_\sigma L \preceq_\pi \mathbf{a}' \\ \mathbf{a} \preceq_\pi \mathbf{a}'' \succeq_\sigma \mathbf{a}'}} \mathbf{a}''.$$

## 6. STREETS.

As before in Section 4, let  $S$  be the set of simple reflections of a finite Coxeter group  $W$ . In this section we relate the conjugation action of the free monoid  $S^*$  on the power set  $\mathcal{P}(S)$  from Section 4 to the takeaway action of  $S^*$  on  $\mathcal{P}(S)$  from Section 5, by extending the conjugation action to the set  $\mathcal{A}$  of all alleys. Again, the  $S^*$ -orbits form a partition of  $\mathcal{A}$ . An orbit of alleys will be called a street and  $\Psi$  will denote the set of all streets. The two main results of this section show that  $\Psi$  is a rooted twofold forest, and that the linear span of  $\Psi$  forms a subalgebra  $\Xi$  of  $\mathcal{A}$ . The partial order on streets allows us to identify a complete set of primitive orthogonal idempotents for  $\Xi$ . We furthermore conjecture that  $\Xi$  is a path algebra.

The conjugate of  $\mathbf{a} = (L; s, t, \dots) \in \mathcal{A}$  by an element  $w \in W$  is the pair

$$(6.1) \quad \mathbf{a}^w = (L^w; s^w, t^w, \dots),$$

consisting of a subset  $L^w \subseteq W$  and a sequence of elements  $s^w, t^w, \dots \in L^w$ . The conjugate  $\mathbf{c}^w$  of an element

$$(6.2) \quad \mathbf{c} = \sum_{\mathbf{a} \in \mathcal{A}} c_{\mathbf{a}} \mathbf{a} \in \mathcal{A}$$



is the linear combination

$$(6.3) \quad \mathbf{c}^w = \sum_{\mathbf{a} \in \mathcal{A}} c_{\mathbf{a}} \mathbf{a}^w,$$

which only is an element of  $\mathcal{A}$  if  $\mathbf{a}^w \in \mathcal{A}$  for all  $\mathbf{a} \in \mathcal{A}$  with  $c_{\mathbf{a}} \neq 0$ . Clearly,  $\mathbf{a}^w \in \mathcal{A}$  if and only if  $L^w \subseteq S$ . Usually we will only consider conjugates  $\mathbf{a}^w$  with  $w \in X_L^\sharp$ . Given  $\mathbf{a}, \mathbf{a}' \in \mathcal{A}$  with  $\iota(\mathbf{a}) = L$  and  $x \in X_L^\sharp$ , by the definition of the partial multiplication on  $\mathcal{A}$ , the product  $\mathbf{a} \circ \mathbf{a}'$  is defined if and only if the product  $\mathbf{a}^x \circ (\mathbf{a}')^x$  is defined, and in that case

$$(6.4) \quad (\mathbf{a} \circ \mathbf{a}')^x = \mathbf{a}^x \circ (\mathbf{a}')^x.$$

The action of the free monoid  $S^*$  on the power set  $\mathcal{P}(S)$ , together with the bijections (4.12) induces an action of  $S^*$  on the set  $\mathcal{A}$  of all alleys as follows. For  $\mathbf{a} = (L; s, t, \dots) \in \mathcal{A}$  and  $r \in S$  we set

$$(6.5) \quad (L; s, t, \dots).r := (L; s, t, \dots)^{\omega(L, r)}.$$

Note that  $(L; s, t, \dots)^{\omega(L, r)} \in \mathcal{A}$  since  $\omega(L, r) \in X_L^\sharp$ . Again, we ignore the obviously trivial edges and define as the *action graph* of this action of  $S^*$  on  $\mathcal{A}$  the graph with vertex set  $\mathcal{A}$  and edge set  $\{\mathbf{a} \xrightarrow{r} \mathbf{a}.r : \mathbf{a} = (L; s, t, \dots) \in \mathcal{A}, r \in S - L\}$ .

**6.1. Proposition.** *Let  $\mathbf{a}, \mathbf{a}' \in \mathcal{A}$  and let  $L = \iota(\mathbf{a})$ . Then  $\mathbf{a}.m = \mathbf{a}'$  for some  $m \in S^*$  if and only if  $\mathbf{a}^x = \mathbf{a}'$  for some  $x \in X_L^\sharp$ .*

*Proof.* Suppose first that  $\mathbf{a}' = \mathbf{a}.m$  for some  $m \in S^*$ . Let  $x = \omega(L, m)$ . Then  $\omega(L, m) \in X_L^\sharp$  by Proposition 4.6 and  $\mathbf{a}' = \mathbf{a}^x$ . Conversely, suppose  $\mathbf{a}' = \mathbf{a}^x$  for some  $x \in X_L^\sharp$ . Let  $(L, m) \in \mathcal{C}$  be a reduced expression for  $(L, x) \in \mathcal{G}$ . Then  $\mathbf{a}' = \mathbf{a}.m$ , as desired.  $\square$

As in Lemma 4.3, if  $\mathbf{a} \in \mathcal{A}$  has  $\iota(\mathbf{a}) = J$  and  $L = J \cup \{s\}$  then

$$(6.6) \quad \mathbf{a}.ss^{w_L} = \mathbf{a}.$$

Hence the effects of  $S^*$  on  $\mathcal{A}$  can be undone and the action of the free monoid  $S^*$  partitions the set of alleys into classes of conjugate alleys. We call such an  $S^*$ -orbit a *street* and denote by

$$(6.7) \quad [\mathbf{a}] = [L; s, t, \dots] = \{\mathbf{a}.m : m \in S^*\}$$

the class of the alley  $\mathbf{a} = (L; s, t, \dots) \in \mathcal{A}$ . Furthermore, we let

$$(6.8) \quad \Psi = \{[\mathbf{a}] : \mathbf{a} \in \mathcal{A}\}$$

denote the set of all streets. The *source* of a street  $[\mathbf{a}]$  is the shape

$$(6.9) \quad \iota([\mathbf{a}]) = [\iota(\mathbf{a})] \in \Lambda$$

and its *target* is the shape

$$(6.10) \quad \tau([\mathbf{a}]) = [\tau(\mathbf{a})] \in \Lambda,$$

both being well defined by Proposition 6.1. The *length* of a street  $[\mathbf{a}]$  is

$$(6.11) \quad \ell([\mathbf{a}]) = \ell(\mathbf{a}).$$

We write

$$(6.12) \quad \mathbf{a} \sim \mathbf{a}'$$

if  $\mathbf{a}, \mathbf{a}' \in \mathcal{A}$  are in the same  $S^*$ -orbit. Note that in general  $\mathbf{a}^w = \mathbf{a}'$  for some  $w \in W$  does not imply  $\mathbf{a} \sim \mathbf{a}'$ . E.g., for the element  $w = sts \in W(A_2) = \langle s, t : s^2 = t^2 = (st)^3 = 1 \rangle$  we have  $(S; s)^w = (S^w; s^w) = (S; t)$ . But  $(S; t)$  is not in the  $S^*$ -orbit of  $(S; s)$  since  $X_S = \{\text{id}\}$  contains no element  $x$  with  $(S; s)^x = (S; t)$ .

$\emptyset \supset \frac{1}{2} \frac{2}{3}$				
$1; 1 \supset_3$ ${}^2 \downarrow \uparrow_1$ $2; 2$ ${}^3 \downarrow \uparrow_2$ $3; 3 \supset_1$	$1 \supset_3$ ${}^2 \downarrow \uparrow_1$ $2$ ${}^3 \downarrow \uparrow_2$ $3 \supset_1$			
$13; 13 \stackrel{2}{\rightleftharpoons} 13; 31$	$13; 1 \stackrel{2}{\rightleftharpoons} 13; 3$	$13 \supset_2$		
$12; 12 \quad 12; 21$ ${}^3 \downarrow \uparrow_1 \quad {}^3 \downarrow \uparrow_1$ $23; 23 \quad 23; 32$	$12; 1 \quad 12; 2$ ${}^3 \downarrow \uparrow_1 \quad {}^3 \downarrow \uparrow_1$ $23; 2 \quad 23; 3$		$12$ ${}^3 \downarrow \uparrow_1$ $23$	
$123; 213 \quad 123; 231$ $123; 123 \quad 123; 132$ $123; 312 \quad 123; 321$	$123; 21 \quad 123; 23$ $123; 12 \quad 123; 13$ $123; 31 \quad 123; 32$	$123; 2$	$123; 3 \quad 123; 1$	$123$

FIGURE 3. Streets of  $A_3$ .

**6.2. Example.** Figure 3 illustrates the streets for  $W$  type  $A_3$  as action graph on the set of all alleys. Here a notation like  $123; 21$  is used as a shorthand for  $(\{1, 2, 3\}; 2, 1) \in \mathcal{A}$ . The streets are arranged in a square grid using their sources and targets as coordinates. (Some entries are highlighted in red in order to illustrate Proposition 10.1.)

For  $L \subseteq S$  and a subset  $\mathcal{B} \subseteq \mathcal{A}$ , the set  $L \circ \mathcal{B}$  is the set of all alleys  $\mathbf{a} \in \mathcal{B}$  with source  $L$  and the set  $\mathcal{B} \circ L$  is the set of all  $\mathbf{a} \in \mathcal{B}$  with target  $L$ . Moreover,

$$(6.13) \quad \mathcal{B} = \bigsqcup_{L \subseteq S} L \circ \mathcal{B} = \bigsqcup_{L \subseteq S} \mathcal{B} \circ L.$$

If, in particular,  $\mathcal{B}$  is a street then even more can be said.

**6.3. Proposition.** *Let  $\mathbf{a} = (L; s, t, \dots) \in \mathcal{A}$ , let  $\lambda = [L]$  and let  $\alpha = [\mathbf{a}]$ . Denote by*

$$N_{\mathbf{a}} = \{\mathbf{x} \in N_L : \mathbf{a}^{\mathbf{x}} = \mathbf{a}\}$$

*the stabilizer of  $\mathbf{a}$ , i.e., the stabilizer of the tuple  $(s, t, \dots)$  in  $N_L$ . Then*

$$\alpha = \bigsqcup_{L' \subseteq S} L' \circ \alpha,$$

*where*

- (i)  $L' \circ \alpha = \emptyset$  unless  $L' \in \lambda$ ;
- (ii) for each  $\mathbf{x} \in X_L^{\#}$ , the map  $\mathbf{a} \mapsto \mathbf{a}^{\mathbf{x}}$  is a bijection from  $L \circ \alpha$  to  $L^{\mathbf{x}} \circ \alpha$ ;
- (iii)  $L \circ \alpha = \mathbf{a}^{N_L}$  is an  $N_L$ -orbit of length  $|N_L : N_{\mathbf{a}}|$ .
- (iv)  $|\alpha| = |\lambda| \cdot |L \circ \alpha| = |\lambda| \cdot |N_L : N_{\mathbf{a}}|$ .

*Proof.* (i) and (ii) are clear. (iii) follows from Proposition 6.1, (iv) from parts (i), (ii) and (iii).  $\square$

With the notation of Proposition 6.3, we call  $|\lambda|$  the *width* of  $\alpha$  and  $|L \circ \alpha|$  the *depth* of  $\alpha$ . Note that, by Proposition 6.3(iii), the depth of a street with source  $L$  is bounded above by the order of the permutation group induced by  $N_L$  on the set  $L$ .

The stabilizer  $N_{\mathbf{a}}$  of an alley  $\mathbf{a} \in \mathcal{A}$  can be described as intersection of normalizer complements.

**6.4. Theorem.** *Let  $\mathbf{a} = (\mathbf{L}; s_1, \dots, s_l) \in \mathcal{A}$ . Then*

$$N_{\mathbf{a}} = N_{\mathbf{L}} \cap N_{\mathbf{L}-\{s_1\}} \cap \dots \cap N_{\mathbf{L}-\{s_1, \dots, s_l\}}.$$

*In particular, if  $\mathbf{a}, \mathbf{a}' \in \mathcal{A}$  are such that the product  $\mathbf{a} \circ \mathbf{a}'$  is defined, then*

$$N_{\mathbf{a} \circ \mathbf{a}'} = N_{\mathbf{a}} \cap N_{\mathbf{a}'}.$$

*Proof.* If  $\ell(\mathbf{a}) = 0$  then  $N_{\mathbf{a}} = N_{\mathbf{L}}$  and there is nothing to prove.

Otherwise,  $\ell(\mathbf{a}) = l > 0$ . Let  $\mathbf{J} = \mathbf{L} - \{s_1, \dots, s_l\}$  and let  $\mathbf{a}' = (\mathbf{L}; s_1, \dots, s_{l-1})$ . By induction on the length  $\ell(\mathbf{a})$ ,

$$N_{\mathbf{a}'} = N_{\mathbf{L}} \cap N_{\mathbf{L}-\{s_1\}} \cap \dots \cap N_{\mathbf{L}-\{s_1, \dots, s_{l-1}\}}.$$

Clearly,

$$N_{\mathbf{a}'} \cap N_{\mathbf{J}} \subseteq \{x \in N_{\mathbf{a}'} : J^x = \mathbf{J}\} = \{x \in N_{\mathbf{a}'} : s_l^x = s_l\} = N_{\mathbf{a}}.$$

Conversely, from  $N_{\mathbf{a}'} \subseteq X_{\mathbf{L}\mathbf{L}} \subseteq X_{\mathbf{L}\mathbf{L}} \subseteq X_{\mathbf{J}\mathbf{J}}$  it follows that

$$N_{\mathbf{a}} = \{x \in N_{\mathbf{a}'} : J^x = \mathbf{J}\} \subseteq N_{\mathbf{a}'} \cap \{x \in X_{\mathbf{J}\mathbf{J}} : J^x = \mathbf{J}\} = N_{\mathbf{a}'} \cap X_{\mathbf{J}\mathbf{J}} = N_{\mathbf{a}'} \cap N_{\mathbf{J}}.$$

Hence

$$N_{\mathbf{a}} = N_{\mathbf{a}'} \cap N_{\mathbf{J}} = N_{\mathbf{L}} \cap N_{\mathbf{L}-\{s_1\}} \cap \dots \cap N_{\mathbf{L}-\{s_1, \dots, s_{l-1}\}} \cap N_{\mathbf{L}-\{s_1, \dots, s_l\}},$$

as desired.  $\square$

**6.5. Products of Streets.** From now on we identify a street  $\alpha \in \Psi$  with the sum  $\sum_{\mathbf{a} \in \alpha} \mathbf{a}$  of all its elements in  $\mathcal{A}$ . Due to the unique factorization property of Proposition 3.1, for each  $\mathbf{b} \in \alpha \circ \alpha'$  there are unique factors  $\mathbf{a} \in \alpha$  and  $\mathbf{a}' \in \alpha'$  such that  $\mathbf{b} = \mathbf{a} \circ \mathbf{a}'$ . Therefore, the product  $\alpha \circ \alpha' \in \mathcal{A}$  coincides with the sum over the set of products  $\{\mathbf{a} \circ \mathbf{a}' : \mathbf{a} \in \alpha, \mathbf{a}' \in \alpha'\}$ . In fact, the product of two streets is a sum of streets, as the next result shows.

**6.6. Theorem.** *Let  $\mathbf{a} = (\mathbf{L}; s, t, \dots) \in \mathcal{A}$  and  $\mathbf{a}' = (\mathbf{L}'; s', t', \dots) \in \mathcal{A}$  be such that  $\mathbf{L}' = \mathbf{L} - \{s, t, \dots\}$ . Let  $\mathbf{D}$  be a set of double coset representatives of  $N_{\mathbf{a}'}$  and  $N_{\mathbf{a}}$  in*

$$N_{\mathbf{L}'} = \bigsqcup_{d \in \mathbf{D}} N_{\mathbf{a}'} d N_{\mathbf{a}}.$$

*Then*

$$[\mathbf{a}] \circ [\mathbf{a}'] = \sum_{d \in \mathbf{D}} [\mathbf{a} \circ (\mathbf{a}')^d].$$

*Proof.* It suffices to show that

$$L \circ [a] \circ [a'] = \sum_{d \in D} L \circ [a \circ (a')^d] = L \circ \sum_{d \in D} [a \circ (a')^d].$$

Then, using Proposition 6.3(i) and (ii), we can conclude that

$$J \circ [a] \circ [a'] = J \circ \sum_{d \in D} [a \circ (a')^d]$$

for all  $J \subseteq S$  and hence the claim follows by equation (6.13).

Using Proposition 6.3(iii), we write

$$(6.14) \quad L \circ [a] \circ [a'] = a^{N_L} \circ [a'] = \sum_{n \in N_L/N_a} a^n \circ [a'],$$

where  $N_L/N_a$  denotes a transversal of the right cosets of  $N_a$  in

$$(6.15) \quad N_L = \bigsqcup_{n \in N_L/N_a} N_a n,$$

and first calculate  $a^n \circ [a']$  for  $n \in N_L$ . Then we form the sum over all these cosets and derive the claimed formula.

Let  $n \in N_L$ . From  $L' \subseteq L$  follows  $N_L \subseteq X_L \subseteq X_{L'}$ , and therefore  $[(a')^n] = [a']$ , by Proposition 6.1. Hence,

$$(6.16) \quad (L')^n \circ [a'] = (L')^n \circ [(a')^n] = (a')^{nN_{(L')^n}} = (a')^{nN_L^n} = (a')^{N_L^n}.$$

We can thus conclude, writing  $a^n = (a \circ L')^n = a^n \circ (L')^n$ , that

$$(6.17) \quad a^n \circ [a'] = a^n \circ (L')^n \circ [a'] = a^n \circ (a')^{N_L^n} = (a \circ (a')^{N_L^n})^n,$$

which shows that

$$(6.18) \quad a^n \circ [a'] = \sum_{n' \in N_L'/N_{a'}} (a \circ (a')^{n'})^n,$$

a sum over the right cosets  $N_{a'}n'$  of  $N_{a'}$  in  $N_{L'}$ .

Let  $D$  be a set of double coset representatives of  $N_{a'}$  and  $N_a$  in  $N_{L'}$ . For each  $d \in D$ , the double coset  $N_{a'}dN_a$  is a union

$$(6.19) \quad N_{a'}dN_a = \bigsqcup_{n'' \in N_a/(N_{a'}^d \cap N_a)} N_{a'}dn''$$

of right cosets of  $N_{a'}$ , parametrized by the right cosets of  $N_{a'}^d \cap N_a$  in  $N_a$ . By Theorem 6.4,

$$(6.20) \quad N_{a'}^d \cap N_a = N_a \cap N_{(a')^d} = N_{a \circ (a')^d}.$$

Moreover, if  $n' \in N_{L'}$ ,  $d \in D$  and  $n'' \in N_a$  are such that  $N_{a'}n' = N_{a'}dn''$  then

$$(6.21) \quad a \circ (a')^{n'} = a^{n''} \circ (a')^{dn''} = (a \circ (a')^d)^{n''}.$$

Therefore, if we denote  $b(d) = a \circ (a')^d$  for  $d \in D$ , then

$$(6.22) \quad \sum_{n' \in N_{L'}/N_{a'}} (a \circ (a')^{n'})^n = \sum_{d \in D} \sum_{n'' \in N_a/N_{b(d)}} b(d)^{n''n}.$$

Now  $\{n''n : n'' \in N_a/N_{b(d)}, n \in N_L/N_a\}$  is a transversal of the right cosets of  $N_{b(d)}$  in  $N_L$ . Thus a summation over  $n \in N_L/N_a$  on both sides finally yields

$$(6.23) \quad \begin{aligned} L \circ [a] \circ [a'] &= \sum_{d \in D} \sum_{m \in N_L/N_{b(d)}} b(d)^m \\ &= \sum_{d \in D} b(d)^{N_L} = \sum_{d \in D} L \circ [b(d)] = L \circ \sum_{d \in D} [b(d)], \end{aligned}$$

as desired.  $\square$

As an immediate consequence, we obtain a product formula for the depths of streets.

**6.7. Corollary.** *Denote by  $\text{dp}(\alpha)$  the depth of  $\alpha \in \Psi$ . Suppose  $\alpha, \alpha' \in \Psi$  and  $\alpha_1, \dots, \alpha_l \in \Psi$  are such that  $\alpha \circ \alpha' = \alpha_1 + \dots + \alpha_l \neq 0$ . Then*

$$\text{dp}(\alpha) \cdot \text{dp}(\alpha') = \text{dp}(\alpha_1) + \dots + \text{dp}(\alpha_l).$$

*In particular, if  $\text{dp}(\alpha) = \text{dp}(\alpha') = 1$  then the product  $\alpha \circ \alpha'$ , if defined, is a single street.*

*Proof.* With the notation from (the proof of) Theorem 6.6, we can derive from

$$(6.24) \quad |N_{L'} : N_{a'}| = \sum_{d \in D} |N_a : N_{a'}^d \cap N_a|$$

that

$$(6.25) \quad \begin{aligned} \text{dp}([a]) \cdot \text{dp}([a']) &= |N_L : N_a| \cdot |N_{L'} : N_{a'}| = \sum_{d \in D} |N_L : N_{a'}^d \cap N_a| \\ &= \sum_{d \in D} |N_L : N_{a \circ (a')^d}| = \sum_{d \in D} \text{dp}([a \circ (a')^d]), \end{aligned}$$

as desired.  $\square$

The action of  $S^*$  on  $\mathcal{A}$  is compatible with the partial orders from Section 5.3, that is with taking prefixes and suffixes. This property can be used to formulate, in Theorem 6.11 below, an alternative product formula for streets in the spirit of Proposition 5.4.

**6.8. Proposition.** *Let  $\mathbf{a} = (L; s, t, \dots) \in \mathcal{A}$  and  $\mathbf{m} \in S^*$ . Then*

- (i)  $\pi(\mathbf{a.m}) = \pi(\mathbf{a}).\mathbf{m}$  and
- (ii)  $\sigma(\mathbf{a.m}) = \sigma(\mathbf{a}).\mathbf{m}'$ , where  $\mathbf{m}' \in S^*$  is such that  $(L_s, \mathbf{m}') \in \mathcal{C}$  is a reduced expression for  $(L_s, \omega(L, \mathbf{m})) \in \mathcal{G}$ .

*Proof.* Let  $\mathbf{d} = \omega(L, \mathbf{m})$ . Then  $\mathbf{a.m} = \mathbf{a}^{\mathbf{d}} = (L^{\mathbf{d}}; s^{\mathbf{d}}, t^{\mathbf{d}}, \dots)$ .

- (i) Clearly,  $\pi(\mathbf{a.m}) = \pi(\mathbf{a}^{\mathbf{d}}) = \pi(\mathbf{a})^{\mathbf{d}} = \pi(\mathbf{a}).\mathbf{m}$ .
- (ii) We have  $\sigma(\mathbf{a.m}) = \sigma(\mathbf{a}^{\mathbf{d}}) = ((L_s)^{\mathbf{d}}; t^{\mathbf{d}}, \dots) = \sigma(\mathbf{a})^{\mathbf{d}}$ , since  $(L^{\mathbf{d}})_{s^{\mathbf{d}}} = (L_s)^{\mathbf{d}}$ . From  $\mathbf{d} \in X_L \subseteq X_{L_s}$  and with Proposition 6.1 it then follows that  $\sigma(\mathbf{a.m}) = \sigma(\mathbf{a}).\mathbf{m}'$ , as desired.  $\square$

In other words, the preimage of any street, under  $\pi$  and under  $\sigma$ , is a union of streets. The streets in the preimage of  $\alpha \in \Psi$  under  $\pi$  can be listed efficiently.

**6.9. Proposition.** *Let  $\mathbf{a} = (L; s_1, \dots, s_k) \in \mathcal{A}$  and let  $J = L - \{s_1, \dots, s_k\}$ . Then*

$$\pi^{-1}([\mathbf{a}]) = \bigsqcup_{\mathbf{t} \in J/N_{\mathbf{a}}} [L; s_1, \dots, s_k, \mathbf{t}],$$

where  $\mathbf{t}$  ranges over a transversal of the  $N_{\mathbf{a}}$ -orbits on  $J$ .

*Proof.* This follows from Proposition 6.3(iii).  $\square$

If we define relations  $\preceq_{\pi}$  and  $\succeq_{\sigma}$  on the set  $\Psi$  of all streets by

$$(6.26) \quad \alpha' \preceq_{\pi} \alpha \text{ if } \alpha' \preceq_{\pi} \mathbf{a} \text{ for some } \mathbf{a} \in \alpha \text{ and some } \mathbf{a}' \in \alpha'$$

and

$$(6.27) \quad \alpha \succeq_{\sigma} \alpha' \text{ if } \alpha' \succeq_{\sigma} \mathbf{a} \text{ for some } \mathbf{a} \in \alpha \text{ and some } \mathbf{a}' \in \alpha'$$

for  $\alpha, \alpha' \in \Psi$ , then both  $\preceq_\pi$  and  $\succeq_\sigma$  are partial orders. The set of streets  $\Psi$  together with the two partial orders  $\preceq_\pi$  and  $\succeq_\sigma$  forms a rooted twofold forest with roots  $\lambda \in \Lambda$ .

n	0	1	2	3	4	5	6	7	8
$A_n$	1	3	8	27	108	536	3180	22113	176175
$B_n$			10	34	136	648	3720	25186	196777
$D_n$					123	579	3417	23387	184580
$E_n$							3347	23057	180570
$F_4$					136				
$H_n$			8	30	120				

TABLE 1. The number of streets for some types of Coxeter groups.

The streets of  $W$  can be efficiently enumerated by using Proposition 6.9 to span the  $\pi$ -forests. In Table 1 we list the number of streets  $|\Psi|$  for some types of Coxeter groups  $W$  of small rank.

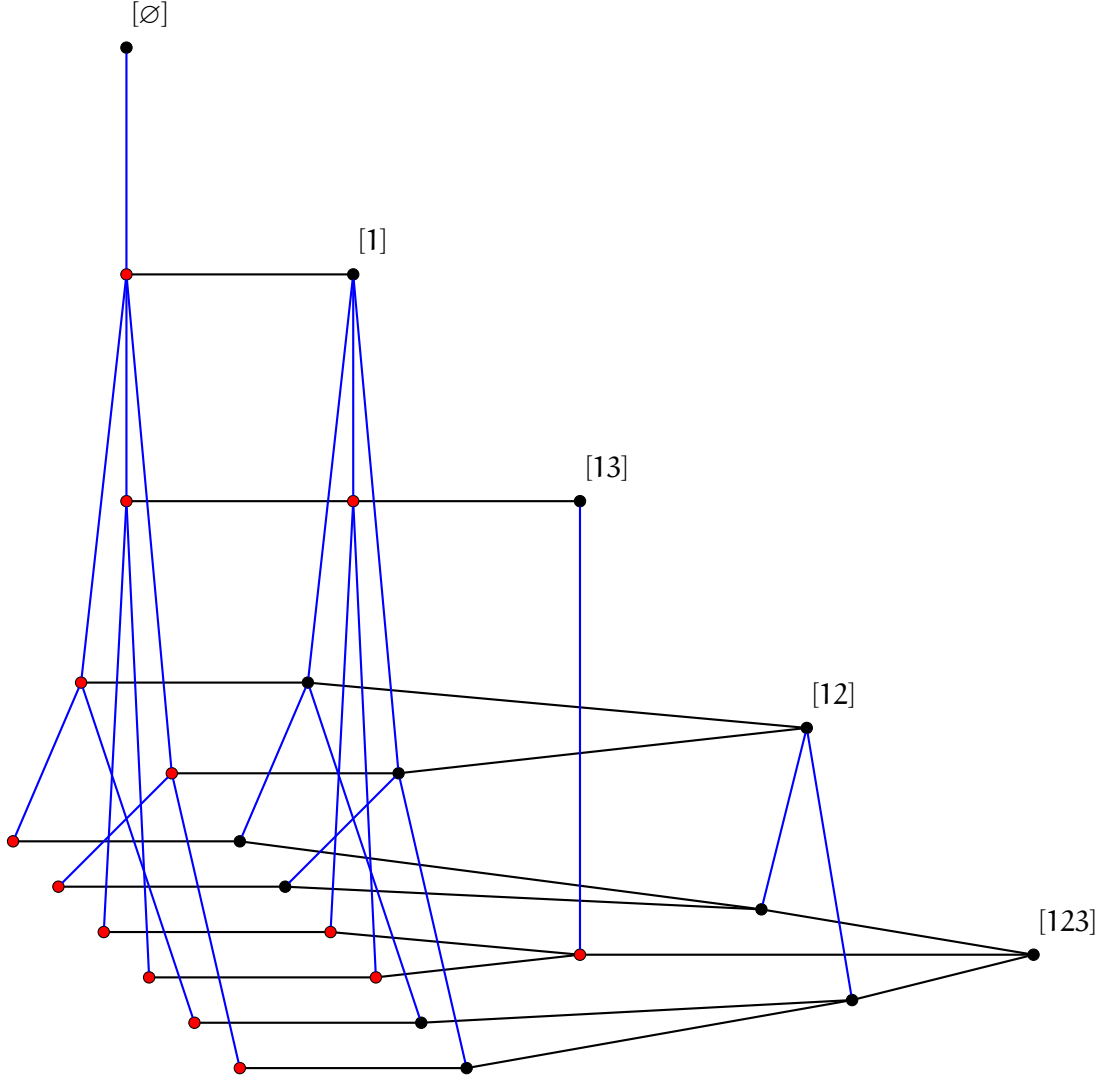
**6.10. Example.** Figure 4 shows the rooted twofold forest for type  $A_3$ . Here the  $\pi$ -forests grow horizontally towards the left and the  $\sigma$ -forests grow vertically downwards out of the roots  $\lambda \in \Lambda$  on the diagonal. The other vertices can be identified with the help of Figure 3. (Some vertices are coloured red in order to illustrate Proposition 10.1.)

The product of  $\alpha, \alpha' \in \Psi$  can be described as the intersection of the  $\pi$ -subtree spanned by  $\alpha$  and the  $\sigma$ -subtree spanned by  $\alpha'$ , provided that  $\tau(\alpha) = \iota(\alpha')$ ; in that case  $\tau(\alpha) = \iota(\alpha')$  is the only shape  $\lambda \in \Lambda$  with  $\alpha \succeq_\sigma \lambda \preceq_\pi \alpha'$ , otherwise there is no such shape at all. In any case, we have the following.

**6.11. Theorem.** *Let  $\alpha, \alpha' \in \Psi$ . Then*

$$\alpha \circ \alpha' = \sum_{\substack{\lambda \in \Lambda, \alpha'' \in \Psi \\ \alpha \succeq_\sigma \lambda \preceq_\pi \alpha' \\ \alpha \preceq_\pi \alpha'' \succeq_\sigma \alpha'}} \alpha''.$$



FIGURE 4. The rooted twofold forest for type  $A_3$ .

*Proof.* By definition,

$$(6.28) \quad \alpha \circ \alpha' = \sum_{\substack{\mathbf{a} \in \alpha, \mathbf{a}' \in \alpha' \\ \tau(\mathbf{a}) = \iota(\mathbf{a}')}} \mathbf{a} \circ \mathbf{a}'.$$

This sum is 0 unless  $\tau(\alpha) = \iota(\alpha')$ . In that case, there is a unique  $\lambda \in \Lambda$  with  $\alpha \succeq_\sigma \lambda$  and  $\lambda \preceq_\pi \alpha'$  and

$$(6.29) \quad \sum_{\alpha \preceq_\pi \alpha'' \succeq_\sigma \alpha'} \alpha'' = \sum_{\alpha \ni \mathbf{a} \preceq_\pi \mathbf{a}'' \succeq_\sigma \mathbf{a}' \in \alpha'} \mathbf{a}'',$$

that is the sum over all  $\mathbf{a}'' \in \mathcal{A}$  such that  $\mathbf{a} \preceq_\pi \mathbf{a}''$  for some  $\mathbf{a} \in \alpha$  and  $\mathbf{a}'' \succeq_\sigma \mathbf{a}'$  for some  $\mathbf{a}' \in \alpha'$ , which is  $\alpha \circ \alpha'$  according to (6.28).  $\square$

**6.12. Corollary.** *Let  $\alpha \in \Psi$ . Then*

- (i) *the elements  $\{\alpha' \in \Psi : \alpha \preceq_\pi \alpha'\}$  form a basis for the right ideal  $\alpha\Xi$  of  $\Xi$ ,*
- (ii) *the elements  $\{\alpha' \in \Psi : \alpha' \succeq_\sigma \alpha\}$  form a basis for the left ideal  $\Xi\alpha$  of  $\Xi$ .*

Moreover, the elements  $\lambda \in \Lambda$  form a complete set of primitive orthogonal idempotents for the algebra  $\Xi$  and the Cartan invariants of  $\Xi$  are

$$\dim \lambda \Xi \lambda' = \#\{\alpha \in \Psi : \lambda \preceq_\pi \alpha \succeq_\sigma \lambda'\},$$

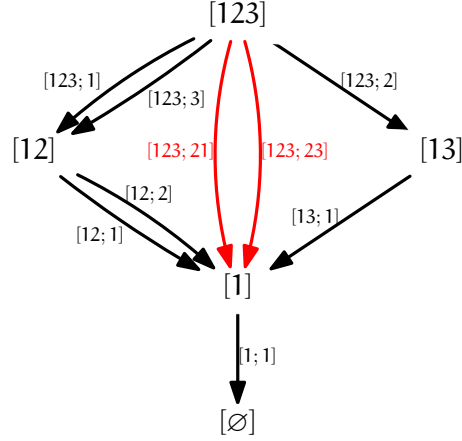
for  $\lambda, \lambda' \in \Lambda$ .

It follows in particular, that the matrix  $(\dim \lambda \Xi \lambda')_{\lambda, \lambda' \in \Lambda}$  of Cartan invariants of  $\Xi$  is unitriangular.

Let us call  $\alpha \in \Psi$  *irreducible*, if  $\ell(\alpha) > 0$  and  $\alpha = \alpha' \circ \alpha''$  for some  $\alpha', \alpha'' \in \Psi$  implies  $\ell(\alpha') \cdot \ell(\alpha'') = 0$ . Then the algebra  $\Xi$  has a presentation as a (not necessarily canonical) quiver with vertex set  $\Lambda$ , edge set corresponding to the irreducible elements of  $\Psi$  and the multiplication table of  $\Psi$  as its only relations.

**6.13. Example.** Figure 5 illustrates the quiver for  $\Xi$  in the case  $A_3$ . There are only two irreducible streets of length greater than 1. The only relevant relation in this case is  $[123; 2] \circ [13; 1] = [123; 21] + [123; 23]$ . This relation allows us to drop one of  $[123; 21]$ ,  $[123; 23]$  from the edge set. Hence  $\Xi$  of type  $A_3$  is in fact a path algebra. The calculations for other types of Coxeter groups suggest that this is always the case.

**6.14. Conjecture.** *For any type of finite Coxeter group, the algebra  $\Xi$  is a path algebra.*

FIGURE 5. A quiver for  $\Xi$  of type  $A_3$ .

## 7. DIFFERENCE OPERATORS.

In this section a grade decreasing difference operator is introduced and shown to eventually map  $\Xi$  surjectively onto the grade 0 component of  $\mathcal{A}$ . In the next section, this difference operator will be used to construct a matrix representation of  $\Xi$ .

For  $i \geq 0$ , denote by

$$(7.1) \quad \mathcal{E}_i = \{a \xrightarrow{r} a.r : a = (L; s, t, \dots) \in \mathcal{A}_i, r \in S - L\}$$

the set of *nontrivial edges* of the graph of the action of  $S^*$  on  $\mathcal{A}_i$ . Then the set of all edges

$$(7.2) \quad \mathcal{E} = \bigsqcup_{i \geq 0} \mathcal{E}_i$$

is in bijection to the set of alleys  $a \in \mathcal{A}$  of positive length.

**7.1. Proposition.** *For each  $i > 0$ , the map*

$$(L; s, t, \dots) \mapsto (L_s; t, \dots) \xrightarrow{s} (L_s; t, \dots).s$$

*is a bijection from  $\mathcal{A}_i$  to  $\mathcal{E}_{i-1}$ .*

In what follows, we will use this bijection to identify edges with elements of  $\mathcal{A}$ . Note that, if  $n = |S|$  then  $\mathcal{A}_n$  has no nontrivial edges, and therefore  $\mathcal{E}_n = \emptyset$ .

We define the *little difference* operator  $\delta: \mathbf{A} \rightarrow \mathbf{A}$  as the linear map which maps an edge  $\mathbf{a} = (\mathbf{L}; \mathbf{s}, \mathbf{t}, \dots) \in \mathcal{A}$  to the difference of its end points, i.e.,

$$(7.3) \quad \delta(\mathbf{a}) = \begin{cases} (\mathbf{L}_s; \mathbf{t}, \dots) - (\mathbf{L}_s; \mathbf{t}, \dots).s, & \text{if } \ell(\mathbf{a}) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then  $\delta(\mathbf{A}_i) \subseteq \mathbf{A}_{i-1}$  for all  $i > 0$ , and hence

$$(7.4) \quad \delta^i(\mathbf{A}_i) \subseteq \mathbf{A}_0$$

for all  $i \geq 0$ . Based on this observation, we furthermore define the *big difference* operator  $\Delta: \mathbf{A} \rightarrow \mathbf{A}_0$  as the linear map with

$$(7.5) \quad \Delta(\mathbf{a}) = \delta^i(\mathbf{a})$$

for  $\mathbf{a} \in \mathbf{A}_i$ . Then, for all  $\mathbf{a} \in \mathcal{A}$ , we have

$$(7.6) \quad \Delta(\mathbf{a}) = \begin{cases} \mathbf{a}, & \text{if } \ell(\mathbf{a}) = 0, \\ \Delta(\delta(\mathbf{a})) & \text{if } \ell(\mathbf{a}) > 0. \end{cases}$$

We will use this latter description of  $\Delta$  for inductive arguments. It also follows that

$$(7.7) \quad 0 = \Delta(\mathbf{a}) - \Delta(\delta(\mathbf{a})) = \Delta(\mathbf{a} - \delta(\mathbf{a})),$$

whence  $\mathbf{a} - \delta(\mathbf{a}) \in \ker \Delta$  for all  $\mathbf{a} \in \mathbf{A}$  with  $\ell(\mathbf{a}) > 0$ .

**7.2. Remark.** The difference operator  $\delta$  does not turn  $\mathbf{A}$  into a chain complex, or a differential graded algebra, since neither  $\delta^2 = 0$  nor the graded Leibniz Rule  $\delta(\mathbf{a} \circ \mathbf{b}) = \delta(\mathbf{a}) \circ \mathbf{b} + (-1)^{\ell(\mathbf{a})} \mathbf{a} \circ \delta(\mathbf{b})$  are satisfied in general.

**7.3. Proposition.** *Let  $\mathbf{a} = (\mathbf{L}; \mathbf{s}, \mathbf{t}, \dots) \in \mathcal{A}$  and let  $\mathbf{x} \in \mathbf{X}_{\mathbf{L}}^{\sharp}$ . Then*

$$(i) \quad \delta(\mathbf{a}^{\mathbf{x}}) = \delta(\mathbf{a})^{\mathbf{x}},$$

$$(ii) \quad \Delta(\mathbf{a}^{\mathbf{x}}) = \Delta(\mathbf{a})^{\mathbf{x}}.$$

*Proof.* (i) From  $\delta(\mathbf{a}) = \sigma(\mathbf{a}) - \sigma(\mathbf{a}).s$  it follows that

$$\begin{aligned} \delta(\mathbf{a}^{\mathbf{x}}) &= \sigma(\mathbf{a}^{\mathbf{x}}) - \sigma(\mathbf{a}^{\mathbf{x}}).s^{\mathbf{x}} \\ &= \sigma(\mathbf{a})^{\mathbf{x}} - \sigma(\mathbf{a})^{\mathbf{x}}.s^{\mathbf{x}} \\ &= \sigma(\mathbf{a})^{\mathbf{x}} - (\sigma(\mathbf{a}).s)^{\mathbf{x}} \\ &= (\sigma(\mathbf{a}) - \sigma(\mathbf{a}).s)^{\mathbf{x}} \\ &= \delta(\mathbf{a})^{\mathbf{x}}, \end{aligned}$$

as desired, since  $\sigma(\mathbf{a}^x) = \sigma(\mathbf{a})^x$  by Proposition 6.8 and  $(\sigma(\mathbf{a}).s)^x = \sigma(\mathbf{a})^x.s^x$ .

(ii) If  $J \subseteq L$  then  $X_L^\# \subseteq X_J^\#$ . Thus, by (i), if  $\mathbf{b} \in \mathcal{A}$  has source  $\iota(\mathbf{b}) = J$  then  $\delta(\mathbf{b}^x) = \delta(\mathbf{b})^x$ . By linearity, this is even true for all  $\mathbf{b} \in \sum_{J \subseteq L} J \circ \mathcal{A}$ . Noting that  $\mathbf{a} \in \sum_{J \subseteq L} J \circ \mathcal{A}$  implies  $\delta(\mathbf{a}) \in \sum_{J \subseteq L} J \circ \mathcal{A}$ , the result follows by applying  $\delta$  sufficiently often.  $\square$

The following graph theoretical lemma will help us to map  $\Xi$  surjectively onto  $\mathcal{A}_0 = \mathbb{Q}[\mathcal{P}(\mathcal{S})]$ .

**7.4. Lemma.** *Let  $(V, E)$  be a connected simple graph with vertex set  $V$  and edge set  $E$ , and let  $X = \mathbb{Q}[V]$  be the vector space with basis  $V$ . Then  $X$  is spanned by  $\sum V$  and the set  $B = \{\mathbf{u} - \mathbf{v} : \{\mathbf{u}, \mathbf{v}\} \in E\}$ .*

*Proof.* Since the graph  $(V, E)$  is connected, for any choice of  $\mathbf{x}, \mathbf{y} \in V$ , there is a path  $(v_0, v_1, \dots, v_l)$  from  $v_0 = \mathbf{x}$  to  $v_l = \mathbf{y}$  in  $(V, E)$ , i.e.  $\{v_{i-1}, v_i\} \in E$  for  $i = 1, \dots, l$ , and  $\mathbf{x} - \mathbf{y} = \sum_{i=1}^l (v_{i-1} - v_i)$  is a linear combination of elements in  $B$ . Now let  $\mathbf{v} \in V$ . Then  $|V|\mathbf{v} = \sum V + \sum_{\mathbf{x} \neq \mathbf{v}} (\mathbf{v} - \mathbf{x})$  and it follows that the space spanned by  $\sum V$  and  $B$  is all of  $X$ .  $\square$

Denote  $\Xi_l = \Xi \cap \mathcal{A}_l$  for  $l \geq 0$ . We conclude this section with the following important result.

**7.5. Theorem.**  $\mathcal{A}_l = \Xi_l \oplus \delta(\mathcal{A}_{l+1})$  for all  $l \geq 0$  and  $\mathcal{A}_0 = \bigoplus_{l \geq 0} \Delta(\Xi_l) = \Delta(\Xi)$ .

*Proof.* Consider the action graph on the vertex set  $\mathcal{A}_l$  for some  $l \geq 0$ . Its edge set corresponds to  $\mathcal{A}_{l+1}$  by Proposition 7.1. The connected components of this graph are the streets  $\alpha \in \Psi_l$ . By Lemma 7.4, the space  $\mathcal{A}_l = \mathbb{Q}[\mathcal{A}_l]$  is spanned by  $\Psi_l \cup \{\delta(\mathbf{a}) : \mathbf{a} \in \mathcal{A}_{l+1}\}$ . Moreover, suppose an element

$$(7.8) \quad \mathbf{c} = \sum_{\mathbf{a} \in \mathcal{A}_l} c_{\mathbf{a}} \mathbf{a} \in \mathcal{A}_l$$

is contained in  $\Xi_l \cap \delta(\mathcal{A}_{l+1})$ . Then  $\mathbf{c} \in \Xi_l$  implies  $c_{\mathbf{a}} = c_{\mathbf{a}'}$  whenever  $\mathbf{a} \sim \mathbf{a}'$ , whereas  $\mathbf{c} \in \delta(\mathcal{A}_{l+1})$  implies

$$(7.9) \quad \sum_{\mathbf{a} \in \alpha} c_{\mathbf{a}} = 0$$

for all  $\alpha \in \Psi_l$ . It follows that  $c_a = 0$  for all  $a \in \mathcal{A}_i$ , whence  $c = 0$  and

$$(7.10) \quad A_l = \Xi_l \oplus \delta(A_{l+1}).$$

It now follows by induction from  $A_{n+1} = 0$  that

$$(7.11) \quad A_0 = \bigoplus_{l \geq 0} \delta^l(\Xi_l) = \bigoplus_{l \geq 0} \Delta(\Xi_l) = \Delta(\Xi),$$

as desired.  $\square$

## 8. A MATRIX REPRESENTATION.

In this section we use the big difference operator  $\Delta$  to turn  $A_0 = \mathbb{Q}[\mathcal{P}(S)]$  into a module for the street algebra  $\Xi$ . This yields an explicit matrix representation  $\mu$  of  $\Xi$ .

For each  $a \in A$ , we define a linear map  $\mu(a) \in \text{End } A_0$  by setting

$$(8.1) \quad L.\mu(a) = \Delta(L \circ a)$$

for all  $L \subseteq S$ .

For an alley  $a = (L; s, t, \dots) \in \mathcal{A}$  this means that  $L.\mu(a) = \Delta(a)$  and  $L'.\mu(a) = 0$  for all  $L' \subseteq S$  with  $L' \neq L$ . For each street  $\alpha \in \Psi$ , we get an endomorphism  $\mu(\alpha)$  of  $A_0$  with the property that

$$(8.2) \quad L.\mu(\alpha) = \sum_{a \in L \circ \alpha} \Delta(a)$$

for all  $L \subseteq S$ . The linear maps  $\mu(\alpha)$ , for  $\alpha \in \Psi$ , have the following crucial property.

**8.1. Proposition.**  $\Delta(a).\mu(\alpha') = \Delta(a \circ \alpha')$  for all  $a \in A$  and all  $\alpha' \in \Psi$ .

*Proof.* It suffices to consider an alley  $a \in \mathcal{A}$ . If  $\ell(a) = 0$  then  $a = \Delta(a) \subseteq S$  and the claim is just equation (8.1).

Otherwise  $\ell(a) > 0$ . Suppose that  $a = (L; s, t, \dots)$ . Let  $d = \omega(L_s, s)$  and  $L' = L - \{s, t, \dots\}$ . Then  $\delta(a) = (L_s; t, \dots) - (L_s; t, \dots)^d$  and  $\Delta(a) = \Delta(\delta(a))$ . By

induction on  $\ell(\mathbf{a})$ ,

$$\begin{aligned}
 (8.3) \quad \Delta((L_s; \mathbf{t}, \dots)).\mu(\alpha') &= \Delta((L_s; \mathbf{t}, \dots) \circ \alpha') \\
 &= \Delta\left(\sum_{\alpha' \in L' \circ \alpha'} (L_s; \mathbf{t}, \dots) \circ \alpha'\right) \\
 &= \sum_{\alpha' \in L' \circ \alpha'} \Delta((L_s; \mathbf{t}, \dots) \circ \alpha').
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 (8.4) \quad \Delta((L_s; \mathbf{t}, \dots)^d).\mu(\alpha') &= \Delta((L_s; \mathbf{t}, \dots)^d \circ \alpha') \\
 &= \Delta\left(\sum_{\alpha' \in L' \circ \alpha'} (L_s; \mathbf{t}, \dots)^d \circ (\alpha')^d\right) \\
 &= \sum_{\alpha' \in L' \circ \alpha'} \Delta(((L_s; \mathbf{t}, \dots) \circ \alpha')^d).
 \end{aligned}$$

Hence

$$\begin{aligned}
 (8.5) \quad \Delta(\mathbf{a}).\mu(\alpha') &= (\Delta((L_s; \mathbf{t}, \dots)) - \Delta((L_s; \mathbf{t}, \dots)^d)).\mu(\alpha') \\
 &= \sum_{\alpha' \in L' \circ \alpha'} \Delta((L_s; \mathbf{t}, \dots) \circ \alpha') - \Delta(((L_s; \mathbf{t}, \dots) \circ \alpha')^d) \\
 &= \sum_{\alpha' \in L' \circ \alpha'} \Delta(\delta(\mathbf{a} \circ \alpha')) \\
 &= \Delta(\mathbf{a} \circ \alpha'),
 \end{aligned}$$

using the facts that  $(L_s - \{\mathbf{t}, \dots\})^d = L_s^d - \{\mathbf{t}^d, \dots\}$  and that  $(\alpha'')^d \circ (\alpha')^d = (\alpha'' \circ \alpha')^d$  by equation (6.4).  $\square$

We denote the restriction of the linear map  $\mu$  to the subalgebra  $\Xi$  of  $\mathbf{A}$  again by  $\mu$ .

**8.2. Theorem.** *The map  $\mu: \Xi \rightarrow \text{End } \mathbf{A}_0$  defined by*

$$L.\mu(\alpha) = \Delta(L \circ \alpha)$$

*is a homomorphism of algebras.*

*Proof.* Let  $\mathbf{a} \in \mathbf{A}$  and  $\alpha' \in \Psi$ . By Proposition 8.1, we have

$$(8.6) \quad L.\mu(\mathbf{a})\mu(\alpha') = \Delta(L \circ \mathbf{a}).\mu(\alpha') = \Delta(L \circ \mathbf{a} \circ \alpha') = L.\mu(\mathbf{a} \circ \alpha'),$$

for all  $L \subseteq S$ . It follows that  $\mu(\alpha \circ \alpha') = \mu(\alpha)\mu(\alpha')$  for all  $\alpha \in A$  and all  $\alpha' \in \Psi$  and thus, in particular,  $\mu(\alpha \circ \alpha') = \mu(\alpha)\mu(\alpha')$  for all  $\alpha, \alpha' \in \Psi$ .  $\square$

## 9. MORE ABOUT DESCENTS.

In this section we identify the  $\Xi$ -module  $A_0$  with the descent algebra  $\Sigma(W)$  and show that the linear maps  $\mu(\alpha)$  for  $\alpha \in \Psi$  are endomorphisms of  $\Sigma(W)$ . It follows that  $\Sigma(W)$  is anti-isomorphic to  $\Xi / \ker \mu = \Xi / \ker \Delta$ . This gives us the desired presentation of  $\Sigma(W)$  as a quiver with relations.

We first take a closer look at certain of the sets  $X_{JKL}$  from equation (2.6).

**9.1. Proposition.** *Let  $s \in S$  and  $J, K, L, M \subseteq S$  be such that  $J \cup \{s\} = L$  and  $K = J^d$ , where  $d = \omega(J, s)$ . Then*

- (i)  $X_{MJJ} \cap X_{MKK} = X_{MLL}$ ;
- (ii)  $(X_{MJJ} - X_{MLL})d = X_{MKK} - X_{MLL}$ .

*Proof.* (i) First note that  $J \subseteq L$  implies  $X_L \subseteq X_J$  and

$$(9.1) \quad X_{MLL} = \{x \in X_M \cap X_L^{-1} : M^x \supseteq L\} \subseteq \{x \in X_M \cap X_J^{-1} : M^x \supseteq J\} = X_{MJJ}.$$

Similarly,  $X_{MLL} \subseteq X_{MKK}$ . Conversely, let  $x \in X_{MJJ} \cap X_{MKK}$ . Then  $x \in X_M$ , and  $x \in X_J^{-1} \cap X_K^{-1} = X_L^{-1}$ , and  $M^x \supseteq J \cup K = L$  whence  $x \in X_{MLL}$ .

(ii) Let  $x \in X_{MJJ} - X_{MLL}$ . By (i) and by symmetry it suffices to show that  $xd \subseteq X_{MKK}$ . We have  $xd \in X_J^{-1}d = X_K^{-1}$  by Proposition 2.1(ii) and  $M^{xd} \cap K = (M^x \cap J)^d = J^d = K$ . It remains to show that  $xd \in X_M$ . We distinguish two cases.

If  $x \in X_L^{-1}$  then  $x \in X_{ML}$  and  $x \in X_{MJJ} - X_{MLL}$  implies  $M^x \cap L = J$ . Hence, using the Mackey decomposition of Proposition 2.1(iii),

$$(9.2) \quad X_M = \bigsqcup_{b \in X_{ML}} bX_{M^b \cap L}^L \supseteq xX_J^L$$

and  $xd \in X_M$ , since  $d = w_J w_L \in X_J^L$ .

Otherwise,  $x \notin X_L^{-1}$ . Then  $s$  is a prefix of  $x^{-1}$  and since  $x^{-1} \in X_J$  and  $J^{x^{-1}} \subseteq M$ , Lemma 4.2(iii) implies that  $d$  is a prefix of  $x^{-1}$ . Therefore,  $xd \in X_M$  as  $xd$  then is a prefix of  $x \in X_M$ .  $\square$



Following Bergeron, Bergeron, Howlett and Taylor [2], we define numbers  $\mathbf{m}_{KL}$ , for  $K, L \subseteq S$ , as

$$(9.3) \quad \mathbf{m}_{KL} = \sum_{J \sim L} \mathbf{a}_{JKL} = \begin{cases} |X_K \cap X_L^\#|, & \text{if } L \subseteq K, \\ 0, & \text{otherwise.} \end{cases}$$

Then, for a suitable ordering of the subsets of  $S$ , the matrix  $(\mathbf{m}_{KL})_{K, L \subseteq S}$  is lower triangular with nonzero diagonal entries

$$(9.4) \quad \mathbf{m}_{KK} = \sum_{J \sim K} \mathbf{a}_{JKK} = \#[K] \cdot |N_K|,$$

for  $K \subseteq S$ , and thus has an inverse  $(\mathbf{n}_{JK})_{J, K \subseteq S}$  over  $\mathbb{Q}$ .

**9.2. Proposition.** *The elements  $\mathbf{e}_J \in \Sigma(W)$ , defined for  $J \subseteq S$  as*

$$(9.5) \quad \mathbf{e}_J = \sum_K \mathbf{n}_{JK} x_K,$$

*form a basis of  $\Sigma(W)$  with*

$$(9.6) \quad \mathbf{e}_J x_M = \sum_{K \sim J} \mathbf{a}_{JMK} \mathbf{e}_K$$

*for all  $J, M \subseteq S$ . Moreover, the elements  $\mathbf{e}_\lambda$ , defined for  $\lambda \in \Lambda$  as*

$$(9.7) \quad \mathbf{e}_\lambda = \sum_{L \in \lambda} \mathbf{e}_L,$$

*form a complete set of primitive orthogonal idempotents of  $\Sigma(W)$  with  $\mathbf{e}_\lambda \mathbf{e}_M = \mathbf{e}_M$  if  $M \in \lambda$ , and  $\mathbf{e}_\lambda \mathbf{e}_M = 0$ , otherwise.*

*Proof.* [2, Theorem 7.8 and Proposition 7.11]. □

We further define, for each alley  $\mathbf{a} = (L; s, t, \dots) \in \mathcal{A}$ , an element  $\mathbf{f}_\mathbf{a} \in \Sigma(W)$  as

$$(9.8) \quad \mathbf{f}_\mathbf{a} = \begin{cases} \mathbf{e}_L, & \text{if } \ell(\mathbf{a}) = 0, \\ \mathbf{f}_{\sigma(\mathbf{a})} - \mathbf{f}_{\sigma(\mathbf{a}).s}, & \text{if } \ell(\mathbf{a}) > 0. \end{cases}$$

More generally, for a linear combination  $\mathbf{c} = \sum_{\mathbf{a} \in \mathcal{A}} c_\mathbf{a} \mathbf{a} \in \mathcal{A}$ , we define

$$(9.9) \quad \mathbf{f}_\mathbf{c} = \sum_{\mathbf{a} \in \mathcal{A}} c_\mathbf{a} \mathbf{f}_\mathbf{a}.$$

Then  $f_a = f_{\delta(a)}$  for all  $a \in \mathcal{A}$  and  $f_c = f_{\Delta(c)}$  for all  $c \in \mathcal{A}$ . In particular, for each street  $\alpha \in \Psi$ , we have an element

$$(9.10) \quad f_\alpha = \sum_{a \in \alpha} f_a,$$

and for each  $L \subseteq S$  we have

$$(9.11) \quad f_{L \circ \alpha} = \sum_{a \in L \circ \alpha} f_a,$$

If we identify  $\mathcal{A}_0$  with  $\Sigma(W)$  by setting  $(L; \emptyset) = e_L$ , we even have  $f_c = \Delta(c)$  for all  $c \in \mathcal{A}$ .

By Proposition 9.2, right multiplication by  $x_M$  maps  $e_J$  to a sum of conjugates of  $e_J$ . The following key result generalizes this property to the elements  $f_a$ . Recall from Section 6 that, if  $a = (L; s, t, \dots) \in \mathcal{A}$  and  $x \in X_L^\sharp$ , then  $a^x = (L^x; s^x, t^x, \dots) \in \mathcal{A}$ .

**9.3. Theorem.** *Let  $L, M \subseteq S$  and  $a = (L; s, t, \dots) \in \mathcal{A}$ . Then*

$$f_a x_M = \sum_{x \in X_{MLL}^{-1}} f_{a^x}.$$

*In particular,  $f_a x_M = 0$  unless  $L$  is contained in a conjugate of  $M$ .*

*Proof.* If  $\ell(a) = 0$  then  $f_a = e_L$  and by equation (9.6),

$$(9.12) \quad e_L x_M = \sum_{K \sim L} a_{LMK} e_K = \sum_{x \in X_{LM} : L^x \subseteq M} e_{L^x \cap M} = \sum_{x \in X_{MLL}^{-1}} e_{L^x},$$

since  $\{x \in X_{LM} : L^x \subseteq M\} = \{x \in X_{ML}^{-1} : M^{x^{-1}} \cap L = L\} = X_{MLL}^{-1}$ .

Otherwise  $\ell(a) > 0$  and  $f_a = f_{(J;t,\dots)} - f_{(K;t',\dots)}$ , where  $J = L_s$ ,  $d = \omega(J, s)$ ,  $K = J^d$  and  $t' = t^d$ . By induction on  $\ell(a)$  and by Proposition 9.1, we have

$$(9.13) \quad \begin{aligned} f_{(J;t,\dots)} x_M &= \sum_{x \in X_{MJJ}^{-1}} f_{(J;t,\dots)^x} \\ &= \sum_{x \in X_{MLL}^{-1}} f_{(J;t,\dots)^x} + \sum_{x \in X_{MJJ}^{-1} - X_{MLL}^{-1}} f_{(J;t,\dots)^x} \end{aligned}$$

and

$$\begin{aligned}
 (9.14) \quad f_{(\mathbf{K}; \mathbf{t}', \dots)} x_M &= \sum_{\mathbf{y} \in X_{M\mathbf{K}\mathbf{K}}^{-1}} f_{(\mathbf{K}; \mathbf{t}', \dots)} \mathbf{y} \\
 &= \sum_{\mathbf{y} \in X_{M\mathbf{L}\mathbf{L}}^{-1}} f_{(\mathbf{K}; \mathbf{t}', \dots)} \mathbf{y} + \sum_{\mathbf{x} \in X_{M\mathbf{J}\mathbf{J}}^{-1} - X_{M\mathbf{L}\mathbf{L}}^{-1}} f_{(\mathbf{J}; \mathbf{t}, \dots)} \mathbf{x}.
 \end{aligned}$$

It follows with Proposition 7.3 that

$$\begin{aligned}
 (9.15) \quad f_{\mathbf{a}} x_M &= (f_{(\mathbf{J}; \mathbf{t}, \dots)} - f_{(\mathbf{K}; \mathbf{t}', \dots)}) x_M \\
 &= \sum_{\mathbf{x} \in X_{M\mathbf{L}\mathbf{L}}^{-1}} (f_{(\mathbf{J}; \mathbf{t}, \dots)} \mathbf{x} - f_{(\mathbf{K}; \mathbf{t}', \dots)} \mathbf{x}) = \sum_{\mathbf{x} \in X_{M\mathbf{L}\mathbf{L}}^{-1}} f_{\mathbf{a} \mathbf{x}},
 \end{aligned}$$

as desired.  $\square$

**9.4. Corollary.** *Let  $\mathbf{a} = (\mathbf{L}; \mathbf{s}, \mathbf{t}, \dots) \in \mathcal{A}$ , let  $\alpha = [\mathbf{a}]$ , let  $\lambda = [\mathbf{L}]$  and let  $M \subseteq S$ . If  $M \in \lambda$  then*

$$(i) \ f_{\mathbf{a}} e_M = \frac{1}{|\alpha|} f_{M \circ \alpha}, \quad (ii) \ f_{L \circ \alpha} e_M = \frac{1}{|\lambda|} f_{M \circ \alpha}, \quad (iii) \ f_{\alpha} e_M = f_{M \circ \alpha}.$$

Otherwise,  $f_{\alpha} e_M = 0$ .

*Proof.* Suppose first that  $M \in \lambda$ . (i) From equation (9.5), we have

$$(9.16) \quad f_{\mathbf{a}} e_M = f_{\mathbf{a}} \sum_J n_{M\mathbf{J}} x_J = \sum_{J \subseteq M} n_{M\mathbf{J}} f_{\mathbf{a}} x_J = n_{MM} f_{\mathbf{a}} x_M = \frac{1}{n_{MM}} f_{\mathbf{a}} x_M$$

since, by Theorem 9.3,  $f_{\mathbf{a}} x_J = 0$  unless  $L$  is contained in a conjugate of  $J \subseteq M$ . Moreover,  $n_{MM} = |N_L| |\lambda|$  and by Theorem 9.3,  $f_{\mathbf{a}} x_M = |N_{\mathbf{a}}| f_{M \circ \alpha}$  since  $M \circ \alpha$  is the  $N_M$ -orbit of  $\mathbf{a} \in \alpha$  and  $N_{\mathbf{a}}$  is its stabilizer. The claim now follows from  $|\alpha| = |N_L : N_{\mathbf{a}}| |\lambda|$ , see Proposition 6.3(iv).

(ii) and (iii) follow easily from (i).

Now suppose that  $M \notin \lambda$ . With  $f_{\lambda} = \sum_{L' \in \lambda} e_{L'}$ , we have

$$(9.17) \quad f_{\alpha} f_{\lambda} = \sum_{L' \in \lambda} f_{\alpha} e_{L'} = \sum_{L' \in \lambda} f_{L' \circ \alpha} = f_{\alpha}.$$

It follows that, for  $M \notin \lambda$ ,

$$(9.18) \quad f_{\alpha} e_M = f_{\alpha} f_{\lambda} e_M = 0,$$

since  $f_{\lambda} e_M = 0$  unless  $M \in \lambda$ .  $\square$

We identify  $A_0$  with  $\Sigma(W)$  by setting

$$(9.19) \quad L = (L; \emptyset) = e_L$$

for all  $L \subseteq S$  and can now formulate the main result of this article.

**9.5. Theorem.** *The linear map  $\Delta: \Xi \rightarrow \Sigma(W)$  defined by  $\alpha \mapsto \Delta(\alpha) = f_\alpha$  for  $\alpha \in \Psi$  is a surjective anti-homomorphism of algebras which induces a bijection between the complete set of primitive orthogonal idempotents  $\lambda \in \Lambda$  of  $\Xi$  and the complete set of primitive orthogonal idempotents  $e_\lambda, \lambda \in \Lambda$ , of  $\Sigma(W)$ .*

*Proof.* Let  $\alpha, \alpha' \in \Psi$  and let  $M \subseteq S$ . Then

$$(9.20) \quad f_\alpha e_M = f_{M \circ \alpha} = \Delta(M \circ \alpha) = M \cdot \mu(\alpha) = e_M \cdot \mu(\alpha).$$

It follows that

$$(9.21) \quad \begin{aligned} f_{\alpha'} f_\alpha e_M &= f_{\alpha'}(e_M \cdot \mu(\alpha)) = (e_M \cdot \mu(\alpha)) \cdot \mu(\alpha') \\ &= e_M \cdot (\mu(\alpha) \mu(\alpha')) = e_M \cdot \mu(\alpha \circ \alpha') = f_{\alpha \circ \alpha'} e_M. \end{aligned}$$

Hence

$$(9.22) \quad f_{\alpha'} f_\alpha = f_{\alpha \circ \alpha'},$$

as desired. □

Note that the linear map defined by  $a \mapsto f_a$  in general is not an algebra homomorphism from  $A$  to  $\Sigma(W)$ : the product of  $(J, \emptyset)$  and  $(K, \emptyset)$  in  $\mathbb{Q}\mathcal{A}$  is zero unless  $J = K$  while  $e_J e_K \neq 0$  if  $J \sim K$  by Proposition 9.2.

As an immediate consequence of Theorem 9.5, we derive some properties of the Cartan matrix of  $\Sigma(W)$  from the Cartan matrix of  $\Xi$ .

**9.6. Corollary.** *The Cartan invariants of  $\Sigma(W)$  are given by the dimensions of the subspaces*

$$e_{\lambda'} \Sigma(W) e_\lambda = \langle f_\alpha : \lambda \preceq_\pi \alpha \succeq_\sigma \lambda' \rangle_{\mathbb{Q}}$$

*of  $\Sigma(W)$ , for all  $\lambda, \lambda' \in \Lambda$ . Hence the matrix  $(\dim e_{\lambda'} \Sigma(W) e_\lambda)_{\lambda', \lambda \in \Lambda}$  of Cartan invariants of  $\Sigma(W)$  is unitriangular.*

*Proof.* By Corollary 6.12, the set  $\{\alpha \in \Psi : \lambda \preceq_\pi \alpha \succeq_\sigma \lambda'\}$  forms a basis of the subspace  $\lambda \Xi \lambda'$  of  $\Xi$ . Under the anti-homomorphism  $\Delta$ , this set is mapped to  $\{f_\alpha : \lambda \preceq_\pi \alpha \succeq_\sigma \lambda'\}$  which therefore spans the subspace

$$e_{\lambda'} \Sigma(W) e_\lambda = \Delta(\lambda \Xi \lambda')$$

of  $\Sigma(W)$ . □

## 10. THE QUIVER OF THE DESCENT ALGEBRA.

Denote by  $\mathbf{Q} = (\mathbf{V}, \mathbf{E})$  the quiver of  $\Sigma(W)$ . This is a graph with vertex set  $\mathbf{V}$  corresponding to the shapes  $\Lambda$  of  $W$ , which, by Proposition 9.2, label the complete set of primitive orthogonal idempotents  $f_\lambda = e_\lambda$ ,  $\lambda \in \Lambda$ , of  $\Sigma(W)$ , and edge set  $\mathbf{E}$  consisting of  $\dim e_{\lambda'}(\text{Rad } \Sigma(W) / \text{Rad}^2 \Sigma(W)) e_\lambda$  edges from  $\lambda'$  to  $\lambda$  for all  $\lambda, \lambda' \in \Lambda$ .

We denote by  $\leq$  the partial order induced on  $\Lambda$  by subset inclusion and by  $\triangleleft$  the cover relation of this partial order, i.e., given  $\lambda, \lambda' \in \Lambda$  we write

$$(10.1) \quad \lambda' \leq \lambda$$

if  $L' \subseteq L$  for some  $L \in \lambda$ ,  $L' \in \lambda'$ , and

$$(10.2) \quad \lambda' \triangleleft \lambda$$

if  $\lambda' < \lambda$  and there is no  $\rho \in \Lambda$  with  $\lambda' < \rho < \lambda$ .

It follows from Corollary 9.6 that, if there is an edge  $\mathbf{e}$  from  $\lambda'$  to  $\lambda$  in the quiver  $\mathbf{Q}$  then  $\lambda' < \lambda$ . Some further properties of  $\Sigma(W)$  and its quiver  $\mathbf{Q}$  follow easily from the description of  $\Sigma(W)$  as anti-homomorphic image of the streets algebra  $\Xi$ . We can, for example, find some streets in  $\ker \Delta$ . Recall from Proposition 7.1 that an alley  $\mathbf{a} = (L; s, t, \dots)$  corresponds to an edge from  $\sigma(\mathbf{a})$  to  $\sigma(\mathbf{a}).s$  in the action graph on  $\mathcal{A}$ . If this edge is a loop then  $\mathbf{a}$  and its  $S^*$ -orbit  $[\mathbf{a}]$  lie in  $\ker \Delta$ . In fact, each street  $\alpha$  which ends in  $[\mathbf{a}]$  then lies in  $\ker \Delta$ , as the next result shows.

**10.1. Proposition.** *Let  $\mathbf{a} = (L; s, t, \dots) \in \mathcal{A}$  be such that  $\ell(\mathbf{a}) > 0$ . If  $\sigma(\mathbf{a}) = \sigma(\mathbf{a}).s$  then  $f_\alpha = 0$  for all  $\alpha \in \Psi$  such that  $\alpha \succeq_\sigma [\mathbf{a}]$ .*

*Proof.* We first consider the case  $\alpha = [\mathbf{a}]$ . From  $\sigma(\mathbf{a}) = \sigma(\mathbf{a}).s$  follows  $\delta(\mathbf{a}) = 0$  and thus  $\Delta(\mathbf{a}) = \Delta(\delta(\mathbf{a})) = \Delta(0) = 0$ . Moreover, by Proposition 7.3, we have  $\Delta(\mathbf{a}.m) = 0$  for all  $m \in S^*$ . Hence  $f_\alpha = \Delta(\alpha) = \sum_{\mathbf{a} \in \alpha} \Delta(\mathbf{a}) = 0$ .

Now suppose  $\alpha = [\mathbf{a}'] \in \Psi$  is such that  $\Delta(\sigma(\mathbf{a}')) = 0$ . Then  $\Delta(\mathbf{a}') = \Delta(\delta(\mathbf{a}')) = 0$  and it follows as before that  $f_\alpha = 0$ . The claim for all  $\alpha \succeq_\sigma [\mathbf{a}]$  then follows by induction.  $\square$

**10.2. Example.** In Figure 3 and Figure 4, for  $W$  of type  $A_3$ , all those  $\alpha \in \Psi$  with  $f_\alpha = 0$  are highlighted in red. In this example, all cases of  $f_\alpha = 0$  can be explained by Proposition 10.1

A street  $\alpha$  may be in  $\ker \Delta$  for other reasons. By equation (6.6), an alley  $\mathbf{a} = (\mathbf{L}; \mathbf{s}, \mathbf{t}, \mathbf{u}, \dots) \in \mathcal{A}$  of length  $l(\mathbf{a}) > 0$  corresponding to the edge  $(\mathbf{L}_s; \mathbf{t}, \mathbf{u}, \dots) \xrightarrow{s} (\mathbf{L}_s; \mathbf{t}, \mathbf{u}, \dots).s$  of the action graph has a reverse edge

$$(10.3) \quad (\mathbf{L}_s; \mathbf{t}, \mathbf{u}, \dots).s \xrightarrow{s'} (\mathbf{L}_s; \mathbf{t}, \mathbf{u}, \dots)$$

going in the opposite direction, where  $s' = s^{w_L}$ . To this reverse edge corresponds the *reverse alley*  $\overline{\mathbf{a}}$  of  $\mathbf{a}$  which we accordingly define as

$$(10.4) \quad \overline{\mathbf{a}} = (\mathbf{L}; s^{w_L}, \mathbf{t}^d, \mathbf{u}^d, \dots),$$

where  $d = \omega(\mathbf{L}_s, s)$ . Clearly,

$$(10.5) \quad \delta(\overline{\mathbf{a}}) = -\delta(\mathbf{a})$$

and

$$(10.6) \quad \overline{\mathbf{a}}^x = \overline{\mathbf{a}^x}$$

for all  $x \in X_L^\sharp$ . From this, the following properties are obvious.

**10.3. Lemma.** *Let  $\alpha \in \Psi$ . Then*

- (i)  $\overline{\alpha} = \{\overline{\mathbf{a}} : \mathbf{a} \in \alpha\} \in \Psi$  with  $\iota(\overline{\alpha}) = \iota(\alpha)$  and  $\tau(\overline{\alpha}) = \tau(\alpha)$ ;
- (ii)  $f_{\overline{\alpha}} = -f_\alpha$ ;
- (iii)  $f_\alpha = 0$  if  $\overline{\alpha} = \alpha$ .

Lemma 10.3(iii) has consequences for the images of streets  $\alpha$  of length  $\ell(\alpha) = 1$  in  $\Sigma(W)$ .

**10.4. Proposition.** *Let  $s \in \mathbf{L} \subseteq \mathbf{S}$ . If there exists a subset  $\mathbf{M}$  with  $\mathbf{L} \subseteq \mathbf{M} \subseteq \mathbf{S}$  such that  $(\mathbf{L}; s)^{w_M} = (\mathbf{L}; s)$  then  $f_{[\mathbf{L}; s]} = 0$ . In particular, if the longest element  $w_0$  is central in  $W$  then  $f_{[\mathbf{L}; s]} = 0$  for all  $\mathbf{L} \subseteq \mathbf{S}$  and  $s \in \mathbf{L}$ .*

*Proof.* The reverse of the alley  $(L; s)$  is  $(L; s^{w_L})$ . Clearly  $x = w_L w_M \in X_L^\#$ . The claim now follows with Lemma 10.3, since  $(L; s^{w_L})^x = (L^{w_M}; s^{w_M}) = (L; s)$ .  $\square$

**10.5. Reduction to Irreducible Finite Coxeter Groups.** It has been shown [2, Proposition 3.2], that if there are subsets  $J, K \subseteq S$  such that  $W$  is the direct product  $W_J \times W_K$  then the descent algebra  $\Sigma(W)$  is the tensor product of  $\Sigma(W_J)$  and  $\Sigma(W_K)$ . The quiver  $\mathbf{Q} = (\mathbf{V}, \mathbf{E})$  of  $\Sigma(W)$  is then the direct product of the quiver  $\mathbf{Q}_J = (\mathbf{V}_J, \mathbf{E}_J)$  of  $\Sigma(W_J)$  and the quiver  $\mathbf{Q}_K = (\mathbf{V}_K, \mathbf{E}_K)$  of  $\Sigma(W_K)$ . It has vertex set  $\mathbf{V} = \mathbf{V}_J \times \mathbf{V}_K$  and edges  $(x', y') \rightarrow (x, y)$  if  $x' = x \in \mathbf{V}_J$  and  $y' \rightarrow y$  is an edge in  $\mathbf{E}_K$  or if  $y' = y \in \mathbf{V}_K$  and  $x' \rightarrow x$  is an edge in  $\mathbf{E}_J$ . Moreover, in every square arising from the product of an edge  $x' \rightarrow x$  in  $\mathbf{E}_J$  and an edge  $y' \rightarrow y$  in  $\mathbf{E}_K$ , the relation

$$(10.7) \quad ((x', y') \rightarrow (x', y) \rightarrow (x, y)) = ((x', y') \rightarrow (x, y') \rightarrow (x, y))$$

holds in  $\Sigma(W)$ .

The problem of finding a quiver presentation for  $\Sigma(W)$  is thus reduced to irreducible Coxeter groups and we will, for the remainder of this section, assume that  $W$  is an irreducible finite Coxeter group. The following property of maximal parabolic subgroups is then easily verified in a case-by-case analysis.

**10.6. Proposition.** *Suppose that  $W$  is an irreducible finite Coxeter group. Let  $L \subseteq S$  and let  $s, s' \in L$  be such that  $L_s \sim L_{s'}$  in  $W$ . Then there exists an  $x \in N_W(W_L)$  such that  $L_s^x = L_{s'}$ , unless  $W$  is of type  $H_3$  and  $W_L$  of type  $A_1 \times A_1$ .*

**10.7. Corollary.** *Suppose that  $W$  is an irreducible finite Coxeter group not of type  $H_3$ . Let  $\lambda, \lambda' \in \Lambda$  be such that  $\lambda' \triangleleft \lambda$  and let  $\alpha, \alpha' \in \Psi$  have common source  $\lambda$  and target  $\lambda'$ . Then  $\alpha' = \alpha$  or  $\alpha' = \bar{\alpha}$ .*

*Proof.* Let  $s, s' \in L \subseteq S$  be such that  $L \in \lambda$  and  $L_s, L_{s'} \in \lambda'$ . Then, by Proposition 10.6,  $L_s^x = L_{s'}$  for some  $x \in N_W(W_L) = W_L \rtimes N_L$ . Let  $J = L_s$  and write  $x = y \cdot z$  for  $y \in X_J^L$  and  $z \in N_L$ . Then  $J$  is a maximal subset of  $L$  and thus either  $y = \text{id}$  or  $y = w_J w_L$ , by Lemma 4.2. With Proposition 6.3(iii) it now follows that either  $(L; s') \sim (L; s)$  or  $(L; s') \sim (L; s^{w_L})$ .  $\square$

**10.8. Remark.** A direct inspection shows that, if  $\lambda, \lambda' \in \Lambda$  are such that  $\lambda' \triangleleft \lambda$  then  $\#\{\alpha \in \Psi : \lambda \preceq_\pi \alpha \succeq_\sigma \lambda'\} \in \{1, 2\}$  holds for  $W$  of type  $H_3$  as well.

Each cover relation gives rise to at most one edge in  $\mathbf{Q}$ .

**10.9. Theorem.** *Let  $\lambda, \lambda' \in \Lambda$  be such that  $\lambda' \triangleleft \lambda$  and denote by  $n_{\lambda\lambda'}$  the number of edges from  $\lambda'$  to  $\lambda$  in  $\mathbf{Q}$ . If there are  $s \in L \subseteq M \subseteq S$  such that  $L \in \lambda$ ,  $L_s \in \lambda'$  and  $(L; s)^{w_M} = (L; s)$  then  $n_{\lambda\lambda'} = 0$ . Otherwise  $n_{\lambda\lambda'} \leq 1$ .*

*Proof.* The number of edges from  $\lambda'$  to  $\lambda$  is given by

$$(10.8) \quad n_{\lambda\lambda'} = \dim e_{\lambda'}(\text{Rad } \Sigma(W) / \text{Rad}^2 \Sigma(W))e_{\lambda} = \dim e_{\lambda'}\Sigma(W)e_{\lambda},$$

since clearly  $e_{\lambda'}\Sigma(W)e_{\lambda} \leq \text{Rad } \Sigma(W)$  and  $e_{\lambda'}\Sigma(W)e_{\lambda} \cap \text{Rad}^2 \Sigma(W) = 0$ . And by Corollary 9.6,  $e_{\lambda'}\Sigma(W)e_{\lambda}$  is spanned by  $\{f_{\alpha} : \lambda \preceq_{\pi} \alpha \preceq_{\sigma} \lambda'\}$ .

If  $W$  is of type  $H_3$  then the longest element  $w_0$  is central in  $W$  and it follows from Proposition 10.4 that  $n_{\lambda\lambda'} = 0$ .

Otherwise, by Corollary 10.7 and Lemma 10.3(ii), the subspace  $e_{\lambda'}\Sigma(W)e_{\lambda}$  is spanned by a single element  $f_{\alpha} = -f_{\bar{\alpha}}$ , which, by Proposition 10.4, is 0 if there are  $s \in L \subseteq M \subseteq S$  such that  $L \in \lambda$ ,  $L_s \in \lambda'$  and  $(L; s)^{w_M} = (L; s)$ .  $\square$

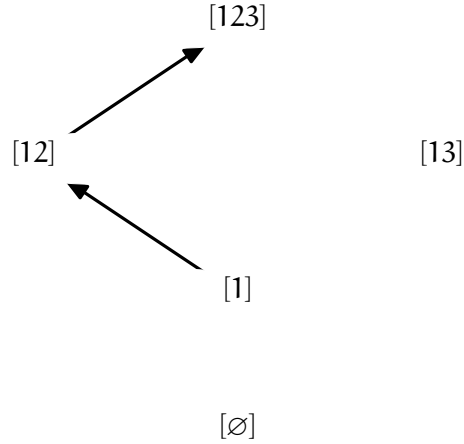


FIGURE 6. The quiver of  $\Sigma(W)$  of type  $A_3$ . There are no relations.

**10.10. Example.** Figure 6 illustrates the case  $A_3$ . The vertices of the quiver are the shapes  $\Lambda$  of  $W$ , which in this case correspond to the partitions of 4. There are only two edges, one mapping to  $f_{[12;1]}$  and the other to  $f_{[123;1]}$  in  $\Sigma(W)$ .



## 11. EXAMPLES OF QUIVER PRESENTATIONS.

In this section we look at particular examples of irreducible finite Coxeter groups. For each of the series  $A_n$ ,  $B_n$  and  $D_n$ , we list some general properties of the quiver  $\mathbf{Q} = (\mathbf{V}, \mathbf{E})$  and give a presentation as a quiver with relations of  $\Sigma(W)$  for the smallest group  $W$  in the series for which the descent algebra is not a path algebra.

**11.1. Algorithm.** Based on Theorem 9.5 and the results of the previous section, we can use the following algorithm to calculate a quiver presentation for the descent algebra  $\Sigma(W)$  of a particular finite Coxeter group  $W$ .

- **Given:** a finite Coxeter group  $W$ .
  - **Compute:** A quiver  $\mathbf{Q} = (\mathbf{V}, \mathbf{E})$  and a set  $\mathcal{R}$  of relations between the paths in  $\mathbf{Q}$  such that the path algebra of  $\mathbf{Q}$  modulo  $\mathcal{R}$  is isomorphic to  $\Sigma(W)$ .
1.  $\mathbf{V} \leftarrow \Lambda$ , the set of all shapes of  $W$ .
  2.  $\mathbf{M} \leftarrow \{\alpha \in \Psi : l(\alpha) > 0 \text{ and } \Delta(\alpha) \neq 0\}$ ;
  3.  $i \leftarrow 0$ .
  4. while  $\mathbf{M} \neq \emptyset$ :
    5.  $i \leftarrow i + 1$ ;
    6.  $\mathbf{E}_i \leftarrow \mathbf{M}$ ;
    7. add to  $\mathcal{R}$  a basis of the nullspace of  $\Delta$  on  $\langle \mathbf{E}_1 \cup \dots \cup \mathbf{E}_i \rangle_{\mathbb{Q}}$ ;
    8. remove redundant elements from  $\mathbf{E}_1$ ;
    9.  $\mathbf{M} \leftarrow \mathbf{M} \circ \mathbf{E}_1$ ;
  10. Return  $(\mathbf{V}, \mathbf{E}_1)$ ,  $\mathcal{R}$  expressed in terms of  $\mathbf{E}_1$ .

In the resulting quiver, the edges are elements of  $\Psi$ , so that an explicit isomorphism between the path algebra of this quiver  $\mathbf{Q}$  and the descent algebra  $\Sigma(W)$  is obtained by simply applying  $\Delta$ .

**11.2. Type  $A_n$ .** The shapes of a Coxeter group of type  $A_n$  correspond to the partitions of  $n+1$ . If  $\lambda, \lambda' \in \Lambda$  are such that  $\lambda' \triangleleft \lambda$  then the partition  $\mathbf{q}$  corresponding to

$\lambda$  is obtained from the partition  $\mathbf{p}$  corresponding to  $\lambda'$  by joining two parts of  $\mathbf{p}$ . And if there is an edge from  $\lambda'$  to  $\lambda$  in  $\mathbf{Q}$  then the two parts are distinct, by Theorem 10.9. It turns out that there is in fact an edge in  $\mathbf{Q}$  whenever the two parts are distinct. It furthermore turns out that  $f_\alpha \in \text{Rad}^2 \Sigma(W)$  for all  $\alpha \in \Psi$  with  $\ell(\alpha) > 1$ .

Hence the vertices of quiver  $\mathbf{Q}$  correspond to the partitions of  $\mathbf{n} + 1$  with an arrow  $\mathbf{p} \rightarrow \mathbf{q}$  between partitions  $\mathbf{p}, \mathbf{q}$  of  $\mathbf{n} + 1$  if and only if  $\mathbf{q}$  is obtained from  $\mathbf{p}$  by joining two distinct parts of  $\mathbf{p}$ . This quiver has an isolated vertex  $1^{n+1}$  and a further isolated vertex  $2^{(n+1)/2}$  if  $\mathbf{n}$  is odd. The remaining vertices form one connected component. The descent algebra  $\Sigma(W)$  therefore has 2 or 3 blocks, depending on whether  $\mathbf{n}$  is odd or even. This description of  $\mathbf{Q}$  has been given by Garsia and Reutenauer [9]. It also follows from the results of Blessenohl and Laue [4, 5], as pointed out by Schocker [18]. A complete proof of this description in the present framework together with a description of the relations in a quiver presentation for  $\Sigma(W)$  of type  $A_n$  will be the subject of a subsequent article.

**11.3. Example.** Consider the Coxeter group  $W$  of type  $A_5$  with Coxeter diagram:

$$1 - 2 - 3 - 4 - 5$$

Here, and similarly in the following examples, we identify the elements of the set  $S = \{1, 2, \dots, 5\}$  with the simple reflections of  $W$ . The vertices  $\mathbf{v} \in \mathbf{V}$  of the quiver

$\mathbf{v}$	type	$\lambda$	$\mathbf{v}$	type	$\lambda$	$\mathbf{v}$	type	$\lambda$
1.	111111	$[\emptyset]$	5.	222	$[135]$	9.	42	$[1235]$
2.	21111	$[1]$	6.	321	$[124]$	10.	51	$[1234]$
3.	2211	$[13]$	7.	411	$[123]$	11.	6	$[S]$
4.	3111	$[12]$	8.	33	$[1245]$			

$\mathbf{e}$	$\alpha$	$\mathbf{e}$	$\alpha$	$\mathbf{e}$	$\alpha$
$2 \rightarrow 4.$	$[12; 1]$	$6 \rightarrow 8.$	$[1245; 1]$	$7 \rightarrow 10.$	$[1234; 1]$
$3 \rightarrow 6.$	$[124; 1]$	$6 \rightarrow 9.$	$[1235; 1]$	$9 \rightarrow 11.$	$[S; 2]$
$4 \rightarrow 7.$	$[123; 1]$	$6 \rightarrow 10.$	$[1234; 2]$	$10 \rightarrow 11.$	$[S; 1]$

TABLE 2. The quiver of  $\Sigma(W)$  for  $W$  of type  $A_5$ .

$\mathbf{Q}$ , correspond to the partitions of 6 and are enumerated in Table 2, together with a representative  $L \subseteq S$  for each shape  $\lambda = [L]$  of  $W$ . The edges  $\mathbf{e} \in \mathbf{E}$  of the quiver  $\mathbf{Q}$

are listed in terms of the vertex numbering and as a streets  $\alpha$ . The only relation in this case is

$$(11.1) \quad (3 \rightarrow 6 \rightarrow 9 \rightarrow 11) = (3 \rightarrow 6 \rightarrow 10 \rightarrow 11),$$

arising from the fact that  $f_{[S;234]} = f_{[S;134]}$  in  $\Sigma(W)$ .

**11.4. Type  $B_n$ .** In the Coxeter group  $W$  of type  $B_n$  the longest element  $w_0$  is central. Proposition 10.4 thus yields  $f_\alpha = 0$  for all  $\alpha \in \Psi$  with  $\ell(\alpha) = 1$ , and therefore no cover relation  $\lambda' \triangleleft \lambda$  of shapes  $\lambda, \lambda' \in \Lambda$  gives rise to an edge of the quiver  $\mathbf{Q}$ !

v	type	$\lambda$	v	type	$\lambda$	v	type	$\lambda$
1.	11111	$[\emptyset]$	11.	321	$[235]$	21.	21	$[1236]$
2.	1111	$[1]$	12.	211	$[125]$	22.	51	$[2345]$
3.	2111	$[2]$	13.	411	$[345]$	23.	11	$[1234]$
4.	211	$[16]$	14.	111	$[123]$	24.	4	$[12456]$
5.	221	$[26]$	15.	32	$[1356]$	25.	5	$[13456]$
6.	311	$[45]$	16.	22	$[1246]$	26.	3	$[12356]$
7.	111	$[12]$	17.	33	$[2356]$	27.	6	$[23456]$
8.	222	$[246]$	18.	31	$[1256]$	28.	2	$[12346]$
9.	221	$[146]$	19.	41	$[1345]$	29.	1	$[12345]$
10.	311	$[145]$	20.	42	$[2456]$	30.	$\emptyset$	$[S]$

e	$\alpha$	e	$\alpha$	e	$\alpha$
$3 \rightarrow 13.$	$[234; 23]$	$9 \rightarrow 28.$	$[12346; 23]$	$12 \rightarrow 29.$	$[12345; 34]$
$3 \rightarrow 14.$	$[123; 12]$	$10 \rightarrow 25.$	$[13456; 34]$	$13 \rightarrow 27.$	$[23456; 23]$
$4 \rightarrow 19.$	$[1345; 34]$	$10 \rightarrow 29.$	$[12345; 23]$	$13 \rightarrow 29.$	$[12345; 12]$
$4 \rightarrow 23.$	$[1234; 23]$	$11 \rightarrow 26.$	$[12356; 12]$	$15 \rightarrow 30.$	$[S; 24]$
$5 \rightarrow 20.$	$[2346; 23]$	$11 \rightarrow 27.$	$[23456; 34]$	$18 \rightarrow 30.$	$[S; 34]$
$5 \rightarrow 21.$	$[1235; 12]$	$11 \rightarrow 27.$	$[23456; 42]$	$19 \rightarrow 30.$	$[S; 23]$
$5 \rightarrow 22.$	$[2345; 34]$	$11 \rightarrow 28.$	$[12346; 12]$	$20 \rightarrow 30.$	$[S; 13]$
$6 \rightarrow 22.$	$[2345; 23]$	$11 \rightarrow 29.$	$[12345; 13]$	$21 \rightarrow 30.$	$[S; 45]$
$6 \rightarrow 23.$	$[1234; 12]$	$12 \rightarrow 24.$	$[12456; 45]$	$22 \rightarrow 30.$	$[S; 12]$
$9 \rightarrow 25.$	$[13456; 45]$				

TABLE 3. The quiver of  $\Sigma(W)$  for  $W$  of type  $B_6$ .

The shapes of  $\mathcal{W}$  correspond to the partitions of  $\mathbf{m} \in \{0, \dots, \mathbf{n}\}$ . Experimental evidence suggests that the edge set  $\mathbf{E}$  of the quiver  $\mathbf{Q}$  on this vertex set can be described as follows. There is an  $\mathbf{e}$ -fold edge  $\mathbf{p} \rightarrow \mathbf{q}$  between two partitions  $\mathbf{p}, \mathbf{q}$  if  $\mathbf{q}$  is obtained from  $\mathbf{p}$  by either joining 3 parts  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  with  $\mathbf{e} = |\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}| - 1$ , or by dropping 2 parts  $\mathbf{a}, \mathbf{b}$  with  $\mathbf{e} = |\{\mathbf{a}, \mathbf{b}\}| - 1$ .

The graph described by these rules has five isolated vertices  $1^{\mathbf{n}}, 1^{\mathbf{n}-1}, 1^{\mathbf{n}-2}, 2^{\lfloor \mathbf{n}/2 \rfloor}$ , and  $3^{\lfloor \mathbf{n}/3 \rfloor}$ , for  $\mathbf{n}$  large enough, and a further isolated vertex  $2^{\mathbf{n}/2-1}$  if  $\mathbf{n}$  is even. The remaining vertices form two connected components, one on the partitions of odd length and one on the partitions of even length. The descent algebra  $\Sigma(\mathcal{W})$  therefore has 4 or 6 blocks if  $\mathbf{n} = 2$  or  $\mathbf{n} = 3$ , and, for larger  $\mathbf{n}$ , it has 8 or 7 blocks, depending on whether  $\mathbf{n}$  is even or odd. The quiver is illustrated with an example of type  $B_6$  below.

We hope to give a complete proof of this description of the quiver together with a description of the relations in a quiver presentation for  $\Sigma(\mathcal{W})$  of type  $B_{\mathbf{n}}$  in a subsequent article.

**11.5. Example.** Consider the Coxeter group  $W$  of type  $B_6$  with Coxeter diagram:

$$1 = 2 - 3 - 4 - 5 - 6$$

The shapes of  $\mathcal{W}$ , which serve as vertex set of the quiver  $\mathbf{Q}$ , correspond to the partitions of all  $\mathbf{m} \in \{0, 1, \dots, 6\}$  and are enumerated in Table 3, together with a representative  $\mathbf{L} \subseteq \mathbf{S}$  for each shape  $\lambda = [\mathbf{L}]$  of  $\mathcal{W}$ . The edges  $\mathbf{e} \in \mathbf{E}$  of the quiver  $\mathbf{Q}$  are listed in terms of the vertex numbering and as a streets  $\alpha$ . Note that there are two edges between vertices 11 and 27, i.e., between the partitions 321 and 6 of 6, which are distinguished by using the symbols  $\dot{\rightarrow}$  and  $\ddot{\rightarrow}$  as arrows.

The only relation in this case is

$$(11.2) \quad (5 \rightarrow 20 \rightarrow 30) = (5 \rightarrow 22 \rightarrow 30),$$

arising from the fact that  $f_{[2346;23]}f_{[S;13]} = f_{[2345;34]}f_{[S;12]}$  in  $\Sigma(\mathcal{W})$ .

**11.6. Type  $D_{\mathbf{n}}$ .** If  $\mathbf{n}$  is even then the longest element  $w_0$  is central in the Coxeter group  $W$  of type  $D_{\mathbf{n}}$  and, as in the case of type  $B_{\mathbf{n}}$ , no cover relation  $\lambda' \triangleleft \lambda$  of shapes  $\lambda, \lambda' \in \Lambda$  gives rise to an edge of the quiver  $\mathbf{Q}$ .

If  $n$  is odd, then the shapes  $\lambda \in \Lambda$  of  $W$  correspond to the partitions of  $m \in \{0, 1, \dots, n-2\} \cup \{n\}$  in such a way that each part  $a$  of a partition  $p$  of  $m$  stands for a direct factor of type  $A_{a-1}$  of  $W_L$ ,  $L \in \lambda$ , and if  $m < n$  then  $W_L$  also has a direct factor of type  $D_{n-m}$ . And if  $\lambda', \lambda \in \Lambda$  correspond to partitions  $p, q$  in this set then  $\lambda' \leq \lambda$  if and only if  $q$  is obtained from  $p$  by either joining two parts or by dropping one part. With Theorem 10.9, it can be shown that, if  $\lambda' \leq \lambda$  and there is an edge from  $\lambda'$  to  $\lambda$  in the quiver  $\mathbf{Q}$ , then  $\lambda'$  corresponds to a partition  $p$  of  $n$ , which has exactly one odd part  $a$ , and the partition  $q$  corresponding to  $\lambda$  arises from  $p$  by either joining  $a$  and another (even) part of  $p$  or, if  $a > 1$ , by dropping  $a$  from  $p$ .

$v$	type	$\lambda$	$v$	type	$\lambda$	$v$	type	$\lambda$
1.	11111	$[\emptyset]$	10.	111	$[123]$	19.	11	$[1234]$
2.	21111	$[1]$	11.	411	$[134]$	20.	4	$[12456]$
3.	1111	$[12]$	12.	22	$[1246]$	21.	3	$[12356]$
4.	2211	$[46]$	13.	31	$[1256]$	22.	2	$[12346]$
5.	3111	$[13]$	14.	33	$[1356]$	23.	$6^-$	$[13456]$
6.	$222^-$	$[146]$	15.	21	$[1236]$	24.	$6^+$	$[23456]$
7.	$222^+$	$[246]$	16.	$42^-$	$[1346]$	25.	1	$[12345]$
8.	211	$[124]$	17.	$42^+$	$[2346]$	26.	$\emptyset$	$[S]$
9.	321	$[236]$	18.	51	$[1345]$			

$e$	$\alpha$	$e$	$\alpha$	$e$	$\alpha$
$2 \rightarrow 10.$	$[123; 12]$	$8 \rightarrow 25.$	$[12345; 34]$	$11 \rightarrow 23.$	$[13456; 13]$
$2 \rightarrow 11.$	$[134; 13]$	$9 \rightarrow 21.$	$[12356; 12]$	$11 \rightarrow 24.$	$[23456; 23]$
$4 \rightarrow 14.$	$[1356; 15]$	$9 \rightarrow 21.$	$[12356; 15]$	$11 \rightarrow 25.$	$[12345; 12]$
$4 \rightarrow 15.$	$[1235; 12]$	$9 \rightarrow 22.$	$[12346; 21]$	$13 \rightarrow 26.$	$[S; 34]$
$4 \rightarrow 16.$	$[1346; 13]$	$9 \rightarrow 22.$	$[12346; 12]$	$14 \rightarrow 26.$	$[S; 41]$
$4 \rightarrow 17.$	$[2346; 23]$	$9 \rightarrow 23.$	$[13456; 34]$	$15 \rightarrow 26.$	$[S; 45]$
$4 \rightarrow 18.$	$[1345; 34]$	$9 \rightarrow 23.$	$[13456; 41]$	$16 \rightarrow 26.$	$[S; 23]$
$5 \rightarrow 18.$	$[1345; 13]$	$9 \rightarrow 24.$	$[23456; 42]$	$17 \rightarrow 26.$	$[S; 13]$
$5 \rightarrow 19.$	$[1234; 12]$	$9 \rightarrow 24.$	$[23456; 34]$	$18 \rightarrow 26.$	$[S; 21]$
$8 \rightarrow 20.$	$[12456; 45]$	$9 \rightarrow 25.$	$[12345; 13]$	$18 \rightarrow 26.$	$[S; 12]$

TABLE 4. The quiver of  $\Sigma(W)$  for  $W$  of type  $D_6$ .

11.7. **Example.** Consider the Coxeter group  $W$  of type  $D_6$  with Coxeter diagram:

$$\begin{array}{c} 1 \\ | \\ 2 - 3 - 4 - 5 - 6 \end{array}$$

The shapes of  $W$ , which serve as vertex set of the quiver  $\mathbf{Q}$ , correspond to the partitions of all  $\mathbf{m} \in \{0, 1, 2, 3, 4, 6\}$ , with two copies of each even partition of 6, and are enumerated in Table 4, together with a representative  $L \subseteq S$  for each shape  $\lambda = [L]$  of  $W$ . The edges  $\mathbf{e} \in \mathbf{E}$  of the quiver  $\mathbf{Q}$  are listed in terms of the vertex numbering and as a streets  $\alpha$ . Note that there are two edges between vertices 9 and 21, 22, 23, 24 respectively, and between vertices 18 and 26.

There are three relations:

$$(11.3) \quad (4 \rightarrow 14 \rightarrow 26) = -2(4 \rightarrow 15 \rightarrow 26),$$

arising from  $f_{[1356;15]}f_{[S;41]} = -2f_{[1235;12]}f_{[S;45]}$  in  $\Sigma(W)$ ,

$$(11.4) \quad (4 \rightarrow 16 \rightarrow 26) = (4 \rightarrow 18 \rightarrow 26),$$

arising from  $f_{[1346;13]}f_{[S;23]} = f_{[1345;34]}f_{[S;21]}$  in  $\Sigma(W)$ , and

$$(11.5) \quad (4 \rightarrow 17 \rightarrow 26) = (4 \rightarrow 18 \rightarrow 26),$$

arising from  $f_{[2346;23]}f_{[S;13]} = f_{[1345;34]}f_{[S;12]}$  in  $\Sigma(W)$ .

11.8. **Exceptional types.** The quivers  $\mathbf{Q}$  of the descent algebras  $\Sigma(W)$  for Coxeter groups  $W$  of exceptional or non-crystallographic type have been computed with Algorithm 11.1 and are described in detail elsewhere [15].

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DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF IRELAND, GALWAY, UNIVERSITY  
ROAD, GALWAY, IRELAND

*E-mail address:* `goetz.pfeiffer@nuigalway.ie`